INTERMEDIATE
TRIGONOMETRY

OUR COLLEGE PUBLICATIONS

INTERMEDIATE
TRIGONOMETRY

BY

B. C. DAS, M. Sc.

PROFESSOR OF MATHEMATICS,
PRESIDENCY COLLEGE, CALCUTTA (RETD.);
EX-LECTURER IN APPLIED MATHEMATICS, CALCUTTA UNIVERSITY

AND

B. N. MUKHERJEE, M.A.

Premchand Roychand Scholar

PROFESSOR OF MATHEMATICS,
SCOTTISH CHURCH COLLEGE, CALCUTTA (RETD.)

TWENTY-SIXTH EDITION

PREFACE TO THE FIRST EDITION

THIS book, as its name indicates, is meant to be a text-book for the Intermediate students of Indian Univer-sities, especially the University of Calcutta. Regarding the subject-matter, we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of examples have been worked out by way of illustrations, and examples set have been carefully selected.

Important formulæ and results have been given at the beginning of the book for reference. Calcutta University questions of recent years are given at the end, to give the students an idea of the standard of the examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labours amply rewarded if the students find the book useful to them.

The book had to be hurried through the press practi-cally within the period of a fortnight, and we must thank the authorities and officers of the K. P. Basu Printing Works, Calcutta, who, in spite of their various preoccu-pations had the kindness to complete the printing in such a short period of time.

Any criticism, correction and suggestion towards improve-ment will be thankfully received.

B. C. D.
B. N. M.

PREFACE TO THE FIFTH EDITION

THIS edition is practically a reprint of the fourth edition ; only a new chapter dealing with harder problems on Heights and Distances, Summations of Finite Trigonometrical Series, and Elimination has been added in the end to cover the syllabuses of some other Indian Universities.

B. C. D.

B. N. M.

PREFACE TO THE TWENTY-FIFTH EDITION

ALTHOUGH this edition is practically a reprint of the previous edition, it contains all those topics that are mentioned in the revised syllabuses of Trigonometry of the Calcutta University and some other Indian Universities.

B. C. D.

B. N. M.

CONTENTS

GREEK LETTERS USED IN THE BOOK

α (Alpha) θ (Thetā)

β (Bētā) π (Pai)

γ (Gamma) ϕ (Phai)

δ (Deltā) ψ (Psi)

Δ (Deltā)

Note. The notation *C. U.* used at the end of any example means that the example was set in Intermediate Examination of the Calcutta University.

TRIGONOMETRY SYLLABUS FOR

INTERMEDIATE EXAMINATIONS

Measurement of angles. Trigonometric ratios and their graphs. Elementary Trigonometrical formulæ. Trigonometric ratios of associated angles. Summation Theorems. Transformation of products and sums. Multiple and sub-multiple angles. Trigonometric Equations and General Values. Inverse Circular Functions. Relation between sides and angles of a triangle. In-radius, Circum-radius and Area of a triangle. In-radius, Circum-radius and Area of a triangle. Partical solutions using log tables. Simple problems in heights and distances.

TRIGONOMETRY SYLLABUS FOR

PRE-UNIVERSITY AND ENTRANCES

Measurement of angles—Sexagesimal and circular measures. Definition of trigonometrical ratios, their mutual relations. Deductions of the values of the trigonometrical ratios of $0°, 30°, 45°, 60°, 90°$. Trigonometrical ratios of associated angle. Addition and subtraction formulæ. Transformation of products and sums of trigonometrical ratios. Multiple and sub-multiple angles (simple cases). General values, Solution of trigonometrical equations. Inverse circular functions. Trigonometrical identities. Relation between sides and angles of a triangle, area, in-radius and circum-radius of a triangle. Solution of triangles with the use of log tables. Graphs of simple trigonometrical functions. Simple problems of heights and distances.

TRIGONOMETRY SYLLABUS FOR

HIGHER SECONDARY COURSE

Class IX

Measurement of angles in degrees, minutes, seconds and in radians. Definition of trigonometrical ratios of an acute angle. Trigonometrical ratios of the standard angles—$0°$, $30°$, $45°$, $60°$, $90°$, (undefined values such as tan $90°$, cot $0°$ to be excluded). Simple identities connecting the ratios of an angle immediately derivable from a right-angled triangle. Trigonometrical ratios of complementary angles.

Easy problems on heights and distances reducible to the solution of right-angled triangles involving the standard angles above.

Class X

Trigonometrical ratios of any angle ; Trigonometrical ratios of angles associated with a given angle ; Addition and subtraction formulæ ; Transformation of products and sums ; Multiple and sub-multiple angles.

Class XI

Graphs of simple trigonometric functions.

Trigonometric equations and general values ; Inverse Circular Functions.

Relation between sides and angles of a triangle ; In-radius, circum-radius and area of a triangle ; Practical solution of a triangle with the help of logarithms ; Simple problems of heights and distances.

IMPORTANT FORMULÆ AND RESULTS

I. A radian $= 57° 17' 44\cdot8''$ nearly.

1 degree $= \cdot01745$ radians nearly.

2 right angles $= 180° = \pi$ radians.

$\pi = \frac{22}{7} = 3\cdot1416$ approximately.

Radian measure of an angle at the centre of a circle
$$= \frac{\text{subtending arc}}{\text{radius}}.$$

II. $\sin^2\theta + \cos^2\theta = 1$; $\dfrac{\sin\theta}{\cos\theta} = \tan\theta.$

$\sec^2\theta = 1 + \tan^2\theta$;

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$ $\dfrac{\cos\theta}{\sin\theta} = \cot\theta.$

III. $\sin 0° = 0$; $\cos 0° = 1$; $\tan 0° = 0.$

$\sin 30° = \dfrac{1}{2}$; $\cos 30° = \dfrac{\sqrt{3}}{2}$; $\tan 30° = \dfrac{1}{\sqrt{3}}.$

$\sin 45° = \dfrac{1}{\sqrt{2}}$; $\cos 45° = \dfrac{1}{\sqrt{2}}$; $\tan 45° = 1.$

$\sin 60° = \dfrac{\sqrt{3}}{2}$; $\cos 60° = \dfrac{1}{2}$; $\tan 60° = \sqrt{3}.$

$\sin 90° = 1$; $\cos 90° = 0$; $\tan 90° = \infty.$

$\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15° = 2 - \sqrt{3}.$

$\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\tan 75° = 2 + \sqrt{3}.$

$\sin 18° = \tfrac{1}{4}(\sqrt{5}-1)$; $\cos 36° = \tfrac{1}{4}(\sqrt{5}+1).$

$\sin 120° = \dfrac{\sqrt{3}}{2}$; $\cos 120° = -\dfrac{1}{2}.$

$\sin 180° = 0$; $\cos 180° = -1$; $\tan 180° = 0.$

$\sin 270° = -1$; $\cos 270° = 0$; $\tan 270° = \infty.$

$\sin 360° = 0$; $\cos 360° = 1$ $\tan 360° = 0.$

IV. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

$\sin(90° - \theta) = \cos\theta$; $\qquad \sin(90° + \theta) = \cos\theta$.

$\cos(90° - \theta) = \sin\theta$; $\qquad \cos(90° + \theta) = -\sin\theta$.

$\tan(90° - \theta) = \cot\theta$; $\qquad \tan(90° + \theta) = -\cot\theta$.

$\sin(180° - \theta) = \sin\theta$; $\qquad \sin(180° + \theta) = -\sin\theta$.

$\cos(180° - \theta) = -\cos\theta$; $\qquad \cos(180° + \theta) = -\cos\theta$.

$\tan(180° - \theta) = -\tan\theta$; $\qquad \tan(180° + \theta) = \tan\theta$.
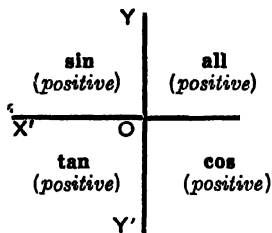
V. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

VI. $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

VII. $\sin C + \sin D = 2 \sin \dfrac{C + D}{2} \ \cos \dfrac{C - D}{2}$

$\sin C - \sin D = 2 \cos \dfrac{C + D}{2} \ \sin \dfrac{C - D}{2}$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}.$$

VIII. $\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}; \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\left.\begin{array}{l} 1 - \cos 2A = 2 \sin^2 A \\ 1 + \cos 2A = 2 \cos^2 A \end{array}\right\}$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}.$$

IX. $\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

X. $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

$$\cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = 2 \cos^2 \frac{\theta}{2} - 1 = 1 - 2 \sin^2 \frac{\theta}{2}$$

$$\tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$$

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}; \quad \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$$

$$1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$$

$$\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{\theta}{2}.$$

XI. If $\sin \theta = \sin a$, then $\theta = n\pi = (-1)^n a$.

If $\cos \theta = \cos a$, then $\theta = 2n\pi \pm a$.

If $\tan \theta = \tan a$, then $\theta = n\pi + a$.

If $\sin \theta = 0$, or, $\tan \theta = 0$, $\theta = n\pi$.

If $\cos \theta = 0$, or, $\cot \theta = 0$, $\theta = (2n+1)\dfrac{\pi}{2}$.

If $\sin \theta = 1$, $\theta = (4m+1)\dfrac{\pi}{2}$; if $\sin \theta = -1$, $\theta = (4m-1)\dfrac{\pi}{2}$.

If $\cos \theta = 1$, $\theta = 2m\pi$; if $\cos \theta = -1$, $\theta = (2m+1)\pi$.

XII. $\sin^{-1}x + \cos^{-1}x = \tfrac{1}{2}\pi$

$\tan^{-1}x + \cot^{-1}x = \tfrac{1}{2}\pi$

$\sec^{-1}x + \operatorname{cosec}^{-1}x = \tfrac{1}{2}\pi$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}$

$\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

$\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\{xy \mp \sqrt{1-x^2}.\sqrt{1-y^2}\}$.

XIII. $\log_a mn = \log_a m + \log_a n$

$\log_a \dfrac{m}{n} = \log_a m - \log_a n$; $\log_a m^n = n \log_a m$;

$\log_a m = \log_b m \times \log_a b$; $\log_a 1 = 0$; $\log_a a = 1$.

XIV. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$;

$\cos B = \dfrac{c^2+a^2-b^2}{2ca}$;

$\cos C = \dfrac{a^2+b^2-c^2}{2ab}$;

$$a = b \cos C + c \cos B.$$
$$b = c \cos A + a \cos C.$$
$$c = a \cos B + b \cos A.$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$$

$$\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ca}$$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab}$$

$$\Delta = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C$$
$$= \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } 2s = a+b+c$$
$$= \frac{abc}{4R}.$$

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}.$$

$$r = \frac{\Delta}{s} = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$
$$= (s-a) \tan \tfrac{1}{2}A = (s-b) \tan \tfrac{1}{2}B = (s-c) \tan \tfrac{1}{2}C.$$

$$r_1 = \frac{\Delta}{s-a} = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$= s \tan \tfrac{1}{2}A.$$

$$r_2 = \frac{\Delta}{s-b} = 4R \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$= s \tan \tfrac{1}{2}B.$$

$$r_3 = \frac{\Delta}{s-c} = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C$$
$$= s \tan \tfrac{1}{2}C.$$

IMPORTANT RESULTS

1. If $A + B + C = \pi$, then

(i) $\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$.

(ii) $\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$.

(iii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

(iv) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

(v) $\cos 2A + \cos 2B + \cos 2C$
$\qquad = -4 \cos A \cos B \cos C - 1$.

(vi) $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$.

(vii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.

(viii) $\sin \dfrac{A}{2} + \sin \dfrac{B}{2} + \sin \dfrac{C}{2}$
$\qquad = 1 + 4 \sin \dfrac{B+C}{4} \sin \dfrac{C+A}{4} \sin \dfrac{A+B}{4}$.

(ix) $\cos \dfrac{A}{2} + \cos \dfrac{B}{2} + \cos \dfrac{C}{2}$
$\qquad = 4 \cos \dfrac{B+C}{4} \cos \dfrac{C+A}{4} \cos \dfrac{A+B}{4}$.

(x) $\tan \dfrac{B}{2} \tan \dfrac{C}{2} + \tan \dfrac{C}{2} \tan \dfrac{A}{2} + \tan \dfrac{A}{2} \tan \dfrac{B}{2} = 1$.

(xi) $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$.

2. $\underset{\theta \to 0}{Lt} \dfrac{\sin \theta}{\theta} = 1$; $\underset{\theta \to 0}{Lt} \cos \theta = 1$; $\underset{\theta \to 0}{Lt} \dfrac{\tan \theta}{\theta} = 1$.

3. Area of a circle of radius $r = \pi r^2$.
Perimeter of a circle of radius $r = 2\pi r$.

TRIGONOMETRY

CHAPTER I

MEASUREMENT OF ANGLES

1. TRIGONOMETRY, as indicated by its very name, originally meant a subject which dealt with the methods of measurement of triangles. At present its scope has widened, and now it means a subject which deals with the measurements relating to any angle, not necessarily an angle of a triangle.

2. Angles in Trigonometry.

In Geometry, angles are supposed to be formed by the intersection of two straight lines and are always restricted to lie between 0° and 360°, being acute, obtuse or reflex. Moreover, they are always positive, negative angles having no meaning. In Trigonometry however, the idea of an angle is much more general.

An angle in Trigonometry is supposed to be formed by the revolution of a straight line which starts from an initial position coinciding with one arm, and traces out the angle by its revolution about one extremity until it reaches the final position coinciding with the other arm.

For instance, the angle XOP is formed by the revolution of a line which starts from the initial position OX, and revolving in the anti-clockwise direction, traces out the angle XOP which is acute. The same line again, starting from OX and revolving in the anti-clockwise direction may make a complete revolution and further move up to the position OQ. The angle formed in this case is more than

five right angles. Now revolutions may be clockwise or anti-clockwise. *It is conventional to consider angles formed by the anti-clockwise revolution of the revolving line to be positive.* Angles formed by clockwise revolutions of the
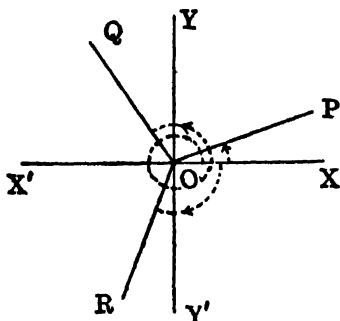
revolving line will then be considered *negative angles.* For example, the angle *XOR* measured in the clockwise direction from the initial position *OX* is a negative angle.

Thus, *angles in Trigonometry may be of any magnitude and may be positive as well as negative.*

OX being the initial position of the revolving line, produce *XO* to *X'*, and let *YOY'* be the perpendicular line. The whole plane is thus divided into four quadrants, the first being *XOY*, the second *YOX'*, the third *X'OY'* and the fourth *Y'OX*. If we contemplate an angle say $+920°$ to be traced out by the revolving line, the line must have completed two complete revolutions, thereby describing $2 \times 360° = 720°$, and have further traced out an angle $200°$, so that the final position of the revolving line is in the third quadrant. Similarly, if we consider an angle $-1354°$, the final position of the revolving line is in the first quadrant, for $-1354° = -360° \times 3 - 274°$.

It should be noted that if two angles differ by complete multiples of $360°$, the starting line being the same, the final

position of the revolving line will be coincident for the two angles. For example, the angles 255° and − 105° will have the final positions of the revolving line same, if both start from the same initial position.

3. Units of measurement of angles.

We should now define the different systems of units used for the measurement of angles. In defining a unit however, a standard angle, which has no reference to any particular system of unit, should form the basis, and such a standard angle is a right angle. A right angle is defined in books on Geometry to be an angle which any straight line standing on another makes with it, when the two adjacent angles formed are equal to one another. A right angle is always the same everywhere, and it thus forms a suitable basis to start with, in defining the different systems of measurement of angles.

There are three systems of units used in Trigonometry for measurement of angles, *viz.*,

(i) **Sexagesimal unit.**

(ii) **Centesimal unit.**

(iii) **Circular unit.**

Sexagesimal* System. In this system, a right angle is divided into 90 equal parts, each being called a *degree*. A degree is again divided into 60 sexagesimal *minutes*, and each minute is further subdivided into 60 sexagesimal *seconds*, so that

1 rt. angle = 90° (degrees)

1°   = 60′ (sexagesimal minutes)

1′   = 60″ (sexagesimal seconds)

---

* So called, since the subdivisions are mostly by sixtieth parts. It is also called the *Common* or the *English System*.

Centesimal† System. In this system, the subdivisions of a right angle are as follows :

1 rt. angle = 100ᵍ (grades)

1ᵍ   = 100' (centesimal minutes)

1'   = 100'' (centesimal seconds).

Note. It may be noted that 1' (centesimal minute) is not the same as 1' (sexagesimal minute), the former being $\dfrac{1}{100 \times 100}$ of a right angle and the latter being $\dfrac{1}{90 \times 60}$ of a right angle, so that the first is $\frac{27}{50}$th part of the second. Similarly, 1'' is less than 1'', being only $\frac{81}{250}$th part of it.

The connection between the two systems of units may be effected through a right angle, remembering that 1 right angle $= 90° = 100^g$, so that $9° = 10^g$. Any angle in the first system may be reduced to degrees, and then multiplied by $\frac{10}{9}$ will be reduced to grades. Similarly, an angle in the second system may be changed to the first.

We shall presently deal with the third system, namely the circular system.

4. Theorem. *In all circles, the circumference bears a constant ratio to its diameter.*

Take any two circles of any radii, and place them with a common centre O. In one, let $ABCD...$ be an inscribed regular polygon of n sides. Let $A', B', C',...$ be the points of intersection of the radii $OA, OB, OC,...$ with the other circle. It is easily seen that $A'B'C'...$ is also a regular polygon of n sides, inscribed in the second circle. Now $OA = OB$ as also $OA' = OB'$, so that in the triangles OAB, $OA'B'$,

---

† So called because the subdivisions are by hundredths. It is also called the *French System*.

$OA : OA' = OB : OB'$, and angle O is common. The two triangles are therefore similar. Hence, $AB : A'B' = OA : OA'$.

Thus,

$$\frac{\text{perimeter of polygon } ABCD\ldots}{\text{perimeter of polygon } A'B'C'D'\ldots} = \frac{n.AB}{n.A'B'} = \frac{OA}{OA'}.$$
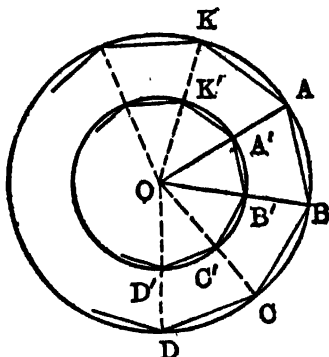
This being true, whatever the number of sides n may be, making n infinitely large, the perimeters of the polygons can be made practically coincident with the circumferences of the corresponding circles, and thus we deduce that

$$\frac{\text{circumference of the circle } ABCD\ldots}{\text{circumference of the circle } A'B'C'D'\ldots} = \frac{OA}{OA'}$$

$$\theta_{\cdot,} = \frac{\text{radius of circle } ABC\ldots}{\text{radius of circle } A'B'C'\ldots},$$

Thus circumference of any circle : its radius is the same for all circles. As diameter is twice the radius, we deduce that the circumference of any circle bears a constant ratio to its diameter.

This constant ratio is denoted by the Greek letter π. Its actual value has been determined by methods which are outside the scope of the present book, by some mathematicians

to more than 500 places of decimals.   An approximate value commonly used is $\frac{22}{7}$.   A more accurate value is $\frac{355}{113}$.

Expressed in decimal, the value is nearly 3·14159...

Hence, if r be the radius of a circle, d its diameter,

the circumference $= \pi d = 2\pi r$,

where $\pi = 3\cdot14159\cdots = \frac{22}{7}$ roughly.

5.   Circular Unit or Radian Measure.

In any circle, if we take an arc whose length is equal to the radius of the circle, the angle which this arc subtends at the centre, is called a *radian*, and is written as 1^c.

We shall now show that with reference to whichever circle it may be defined, a radian is a constant angle, and hence it may be used as a suitable unit for measurement of angles, which is known as the circular unit.

Theorem I.   *A radian is a constant angle.*
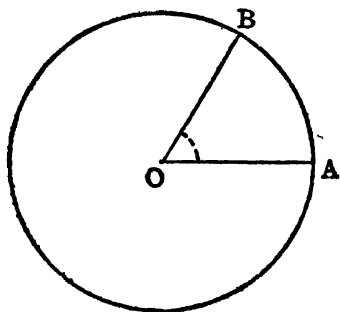
Let AB be an arc of any circle with centre O, whose length is equal to its radius OA. By definition, $\angle AOB =$ 1 radian.   Since angles at the centre of a circle are proportional to the arcs which subtend them, and the whole angle

round O subtended by the complete circumference being known from Geometry to be 4 right angles, we get

$$\frac{\angle AOB}{4 \text{ right angles}} = \frac{\text{arc } AB}{\text{whole circumference}} = \frac{\text{radius}}{\text{circumference}},$$

i.e., $\quad \dfrac{1 \text{ radian}}{4 \text{ rt. } \angle^s} = \dfrac{r}{2\pi r} = \dfrac{1}{2\pi},$ r being the radius.

Hence, 1 radian $= \dfrac{2}{\pi}$ rt. angle.

∴ a radian is a constant angle (π being constant).

Note. We thus see that whatever be the radius of the circle with reference to which a radian is defined, its magnitude is the same.

From above, π radians $= 180°$.

$$\therefore \quad 1 \text{ radian} = \frac{180}{\pi} = \frac{180}{3\cdot14159} = 57\cdot29577 \text{ degrees}$$

$$= 57° \ 17' \ 44\cdot8'' \text{ nearly,}$$

∴ **1 degree** $= \cdot0174533$ **radians** nearly.

In higher mathematics so far as theoretical investigations are concerned, as a matter of convenience, angles are usually measured in the circular unit, *i.e.*, in radians. In this connection we may state the following theorem :

Theorem II. *The measure of any angle in radians is expressed by the ratio of the arc of any circle subtending that angle at its centre, to the radius.*

Let XOP be any angle.

With centre O and any radius OA draw a circle, and let AQ be the arc which subtends the angle XOP at the centre O. Let AB be the arc whose length is equal to the radius AO, so that, by definition, $\angle AOB$ is one radian.

Now from Geometry, angles at the centre of a circle are proportional to the arcs which subtend them.

Hence, $\dfrac{\angle XOP}{\angle AOB} = \dfrac{\text{arc } AQ}{\text{arc } AB} = \dfrac{\text{arc } AQ}{\text{radius } OA}$,

or, $\dfrac{\angle XOP}{1 \text{ radian}} = \dfrac{\text{arc } AQ}{\text{radius } OA}$,

i.e., $\angle XOP = \dfrac{\text{arc } AQ}{\text{radius } OA}$ of a radian.
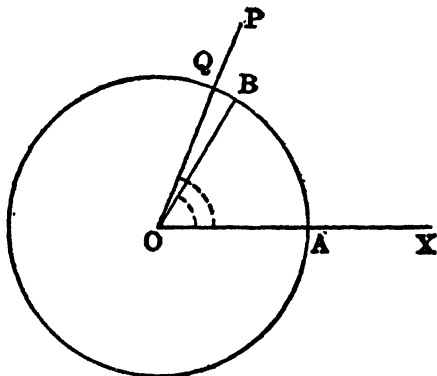
Thus, if θ be the radian-measure of the $\angle XOP$, s be the length of the arc AQ, and r the radius of the circle, then

$$\theta = \dfrac{s}{r} \text{ or, } s = r\theta.$$

Note. In higher mathematics, *when an angle is expressed in radian-measure, the unit is generally implied and not expressed,* so that, when the measure of an angle is given without the unit being mentioned, we should always understand it to be in radians. For example, 'an angle is $\dfrac{\pi'}{2}$' means that the angle is $\dfrac{\pi}{2}$ radians, which converted to degrees is $90°$ i.e., one right angle.

6. In working out examples, relations between the three systems of units should be carefully remembered, namely

$$1 \text{ rt.} \angle = 90° = 100^g = \dfrac{\pi}{2} \text{ radians,}$$

whence, $\pi^c = 180°$.

Ex. 1. *Express*

    (i) 63° 22′ 40·8″ *in centesimal measure*

and (ii) 203^g 58′ 73″ *in radians.*

Here, (i) $63° 22′ 40·8″ = 63\frac{189}{500}$ deg. $= \frac{31689}{500} \times \frac{1}{90}$ rt. $\angle$

$$= \frac{31689}{500} \times \frac{1}{90} \times 100 \text{ grades} = \frac{7042}{100} \text{ grades}$$

$$= 70^g\ 42′.$$

    (ii) $203^g\ 58′\ 73″ = 203·5873$ grades

$$= 2·035873 \text{ rt.} \angle = 2·035873 \times \frac{\pi}{2} \text{ radians}$$

$$= 1·0179365\pi \text{ radians.}$$

Ex. 2. *Two angles of a triangle are* 72° 53′ 51″, *and*
41^g 22′ 50″ *respectively. Find the third angle in radians.*

$$41^g\ 22′\ 50″ = 41·2250 \text{ grades}$$

$$= \frac{41·225 \times 9}{10} \text{ degrees } [\ 9° = 10^g\ ]$$

$$= 37·1025 \text{ degrees}$$

$$= 37°\ 6′\ 9″.$$

The sum of the two given angles is therefore
$$72°\ 53′\ 51″ + 37°\ 6′\ 9″ = 110°.$$

The sum of the three angles of a triangle being 180°, the
third angle is

$$180° - 110° = 70° = 70° \times \frac{\pi}{180} \text{ radians } [\ \pi^c = 180°\ ]$$

$$= \frac{7\pi}{18} \text{ radians.}$$

Ex. 3. *Divide* $\frac{\pi}{4}$ *radians into two parts such that the
number of sexagesimal minutes in one may be to the number
of centesimal seconds in the other part as* 27 : 2500.

We have $\frac{\pi}{4}$ radians $= \frac{\pi}{4} \times \frac{2}{\pi}$ rt. $\angle = \frac{1}{2}$ rt. $\angle$.

Let x be the number of centesimal seconds in the second part, so that $\frac{27}{2500}x$ is the number of sexagesimal minutes in the first part.

Now, $x'' = \dfrac{x}{100 \times 100 \times 100}$ rt. $\angle$

and $\dfrac{27}{2500}x' = \dfrac{27x}{2500 \times 60 \times 90}$ rt. $\angle = \dfrac{x}{500000}$ rt. $\angle$,

$\therefore \dfrac{x}{1000000} + \dfrac{x}{500000} = \dfrac{1}{2}$,

whence $x = \dfrac{500000}{3}$.

Thus, second part is $\dfrac{500000''}{3} = \dfrac{500000}{3 \times 100 \times 100 \times 100}$ rt. $\angle$

$= \frac{1}{6}$ rt. $\angle = 15°$, and as the sum of the two parts is $\frac{1}{2}$ rt. $\angle$ i.e., $45°$, the first part is $30°$.

The two parts are therefore $30°$ and $15°$.

Ex. 4. *The angles of a quadrilateral are in A.P., and the number of grades in the least angle is to the number of radians in the greatest as $100 : \pi$. Find the angles in degrees.*

Let the angles, expressed in degrees, be a, $a + \beta$, $a + 2\beta$ and $a + 3\beta$ respectively.   Then

$a + a + \beta + a + 2\beta + a + 3\beta = 360,$ $\cdots$ (1)

i.e., $2a + 3\beta = 180.$

Again the least angle, $a° = \frac{10}{9}a^g$

and the greatest angle, $(a + 3\beta)° = (a + 3\beta)\,\dfrac{\pi^c}{180}$,

and so from the given condition,

$\dfrac{10}{9}\,a/(a + 3\beta)\,\dfrac{\pi}{180} = 100/\pi,$

or, $\dfrac{2a}{a + 3\beta} = 1$, whence $a = 3\beta$.

∴   using (1), $3a = 180$, or, $a = 60$ and $\beta = \dfrac{a}{3} = 20$.

Thus the angles are

$$60°, \ 80°, \ 100° \text{ and } 120°.$$

Ex. 5. *At what distance does a man, $5\frac{1}{2}$ ft. in height subtend an angle of $15''$?*

B

$5\frac{1}{2}$ ft.

$15''$

O

r ft.

A
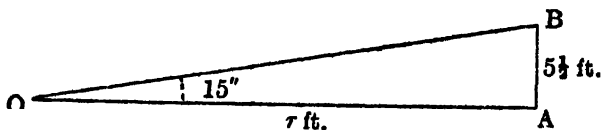
AB being the man subtending an angle $15''$ at O, let OA be r ft.

As the angle AOB is very small, so that AB is very small compared to AO, we may assume the small length AB to be practically a small arc of a circle whose centre is O. Now the measure of an angle in radians is the ratio of the arc which subtends it at the centre to the radius.

[See *Art. 5*]

$$\therefore \quad \frac{15}{60 \times 60} \times \frac{\pi}{180} = \frac{5\frac{1}{2}}{r},$$

$$\text{or,} \quad r = \frac{11}{2} \times \frac{180 \times 60 \times 60}{15 \times \pi} \text{ ft.}$$

$$= \frac{11}{2} \times \frac{180 \times 60 \times 60 \times 7}{15 \times 22} \times \frac{1}{3 \times 1760} \text{ miles approx.}$$

$$= 14\text{·}32 \text{ miles nearly.}$$

Examples I

1. Indicate the final position of a revolving line which has traced out the angle

(i) $1122°$;   (ii) $-810° \ 29'$;

(iii) $-617^g \ 51' \ 5''$; , (iv) $\dfrac{18\pi}{5}$ radians.

2. Express (i) $55°$ $12'$ $36''$ in centesimal measure ;
                    (ii) 195^g $35'$ $24''$ in degrees, minutes, and secs.

3. How many radians are there in (i) 50^g $75'$ $50''$;
(ii) $18°$ $33'$ $45''$?

4. Express in each system of angular measurement the angle between the minute-hand and the hour-hand of a clock at quarter to twelve.

5. If x^g be taken as the unit angle, and the angles $600°$ and 16^c expressed in that unit be a and β respectively, find the relation between a and β.

6. The difference of two angles is $1°$; the circular measure ôf their sum is 1 ; find the circular measure of the smaller angle.

7. Two angles are in the ratio $2 : 3$, and the difference of their measure in grades and in degrees respectively is $2\frac{1}{2}$; find the angles in degrees.

8. An angle is the excess of $D°$ M' over G^g m'. Find the ratio of this angle to a right angle.

9. The circular measure of a certain angle is equal to the ratio of the number of degrees in it to the number of centesimal minutes ; find the magnitude of the angle in degrees.

10. With two units of angular measurements differing by $10°$, the measure of an angle are as $3 : 2$; determine the units.

11. If an angle standing upon an arc of length 'l' at the centre of a circle of radius 'r' be taken as unit, and three angles $D°$, G^g, and C circular units expressed in that unit be x, y, z respectively, show that

$$x : y : z = \frac{D\pi}{18} : \frac{G\pi}{20} : 10C.$$

12. Three angles are in G.P. The number of grades in the greatest angle is to the number of circular units in the least as 800 to π, and the sum of the three angles is 126°. Find the angles in grades.

13. Divide 54° in three parts, such that the circular measure of the first exceeds that of the second by $\frac{\pi}{10}$, and the sum of the second and third is 30 grades.

14. Find at what times between 7 and 8 o'clock the angle between the two hands of a clock is (i) 60^g, (ii) 155°.

15. The angles of a triangle are in A.P., and the number of radians in the greatest is to the number of grades in the least as π : 40. Find the angles in degrees.

16. In each of two triangles the angles are in G.P.; the least angle of one of them is three times the least angle in the other, and the sum of the greatest angles is 240°. Find the circular measure of the angles.

17. One angle of a quadrilateral is $\frac{8}{9}$ of another and the two other angles are 66$\frac{2}{3}$ grades and $\frac{3}{4}\pi$ radians. Express the angles in degrees.

18. The angles of a polygon (which has no reflex angle) are in A.P. The least angle is $\frac{2}{3}\pi$ radians and the common difference is 5°. Find the number of sides.

19. The number of sides of two regular polygons are as m : n and the number of degrees in an angle of the first is to the number of grades in an angle of the second as p : q. Determine the number of sides in each polygon.

20. An arc of 50^g in one circle equals one of 60° in another; find the radian-measure of an angle subtended at the centre of the first circle by an arc equal to the radius of the second.

21. Two regular figures are such that the number of degrees in an angle of one is to the number of degrees in an angle of the other as the number of sides in the first is to the number of sides in the second. The sum of the number of sides of the two figures being 9, determine the number of sides of each.

22. The wheel of a railway carriage is 4 ft. in diameter and makes 6 revolutions in a second ; how fast is the train going ?

23. The earth revolves round the sun in a circular orbit of radius 92700000 miles once a year. Find its velocity in miles per hour. If the apparent angular diameter of the sun observed from the earth be 32′, find also the linear radius of the sun.

24. A tower subtends an angle of 10′ when the observer is at a distance of 6 miles ; find its height.

25. Find the radius of the earth, if an angle of 1° is subtended at its centre by an arc joining two places on it distant 69˙1 miles.

26. A horse is tied to a post by a rope 27 feet long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far the horse will have gone when the rope has traced out an angle of 70°. ($\pi = \frac{2\cdot2}{7}$)

27. A man running along a circular track at the rate of 10 miles per hour, traverses in 36 seconds, an arc which subtends 56° at the centre. Find the diameter of the circle. ($\pi = \frac{2\cdot2}{7}$)

28. An arc of 30° in one circle is double an arc in a second circle, the radius of which is three times the radius of the first. Show that the arc of the second circle subtends 5° at its centre.

TRIGONOMETRICAL RATIOS*

7. Trigonometrical ratios defined.
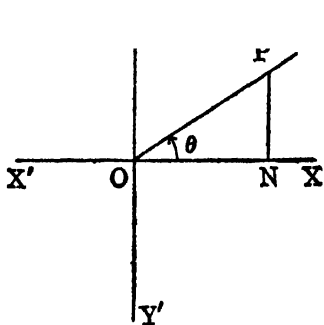
Fig. 1
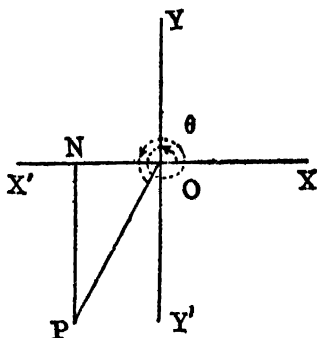
Fig. 2

Let θ be the measure of an angle XOP which may be supposed to be traced out by a revolving line starting from the initial position OX. From any point P on its other arm, draw a perpendicular PN on OX (produced if necessary, as in the second figure). A right-angled triangle is thereby formed. The trigonometrical ratios of the angle θ are defined as follows :—

Sine of the angle θ, written as $\sin \theta = \dfrac{PN}{OP}$

$$i.e., \quad \frac{opposite\ side}{hypotenuse}$$

Cosine of θ, written as $\cos \theta = \dfrac{ON}{OP}$

$$i.e., \quad \frac{adjacent\ side}{hypotenuse}$$

---

* For alternative definitions, see *Appendix*.

Tangent of θ, written as $\tan \theta = \dfrac{PN}{ON}$

$$i.e., \quad \frac{opposite\ side}{adjacent\ side}$$

Cosecant of θ, written as $\text{cosec } \theta = \dfrac{OP}{PN}$

$$i.e., \quad \frac{hypotenuse}{opposite\ side}$$

Secant of θ, written as $\sec \theta = \dfrac{OP}{ON}$

$$i.e., \quad \frac{hypotenuse}{adjacent\ side}$$

Cotangent of θ, written as $\cot \theta = \dfrac{ON}{PN}$

$$i.e., \quad \frac{adjacent\ side}{opposite\ side}$$

In addition to these, we define two less important ratios of the angle θ which are sometimes used, as follows :—

Versed sine of angle θ, written as **vers** $\theta = 1 - \cos \theta$.

Coversed sine of angle θ, written as **covers** $\theta = 1 - \sin \theta$.

8. Signs of Trigonometrical ratios.

XOP being any angle, traced out by a revolving line which starts from OX, it has already been mentioned in the previous Chapter that the plane may by divided into four quadrants by the two perpendicular lines XOX' and YOY', the first being horizontal and the second vertical.

It is conventional, as in graphs, to consider distances measured along OX and OY as positive, and along OX' and OY' as negative. *The distance measured along OP*, the final position of the revolving line corresponding to the angle XOP, in whichever quadrant it may lie, *is however always considered positive.*

With this convention, if OP lies in the first quadrant as in Fig. (i) of the last article, the sides PN, ON and OP of the right-angled triangle OPN are all positive. Hence, all the trigonometrical ratios are positive. If OP lies in the third quadrant as in Fig. (ii), ON and PN are both negative, but OP is positive. Hence, from the definition of the trigonometrical ratios, sin XOP $\left(-\dfrac{PN}{OP}\right)$ is negative, cos XOP $\left(-\dfrac{ON}{OP}\right)$ is negative, tan XOP $\left(-\dfrac{PN}{ON}=\dfrac{\text{negative quantity}}{\text{negative quantity}}\right)$ is positive etc.

In this way, according to the final position of the revolving line (starting position being OX), we can determine the signs of the trigonometrical ratios of the angle XOP whether this angle traced out is positive or negative. If OP is in the *first quadrant, the ratios are all positive.* If OP falls in the *second quadrant, sine and cosecant* (which is evidently the reciprocal of sine), *are positive* ; all the other ratios are negative. If OP be in the *third quadrant, tangent and cotangent* (which are reciprocals to each other) *are positive* ; all the others are negative. In the *fourth quadrant, cosine and secant are positive,* others are negative. A symbolical figure will help the memory in this case, namely that according to the position of OP,

	Y	
sin (*positive*)		**all** (*positive*)
X'	O	X
tan (*positive*)		**cos** (*positive*)
	Y'	
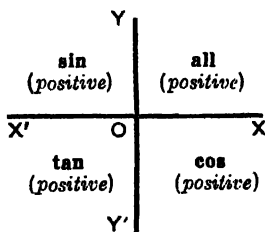
The positiveness of sine, cosine and tangent also implies the positiveness of their reciprocals namely, cosecant, secant and cotangent respectively.

9. Constancy of Trigonometrical ratios.

So long as an angle remains the same, its trigonometrical ratios are unique.
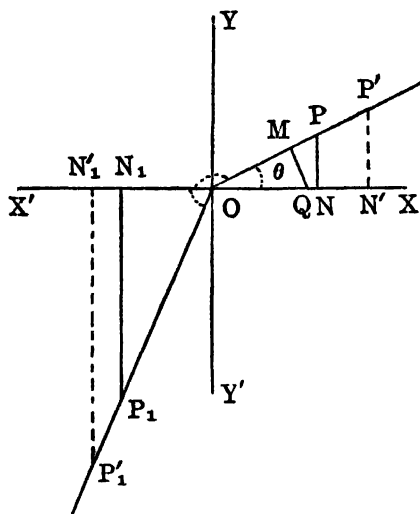
Let $XOP\ (=\theta)$ be any angle, and let PN and $P'N'$ be drawn perpendiculars upon OX from any two points P and P' on OP. The two right-angled triangles OPN and $OP'N'$ are similar. Hence, $\sin\theta$, whether we take it as $\dfrac{PN}{OP}$ or $\dfrac{P'N'}{OP'}$ is the same. If the angle be XOP_1, when OP_1 is not in the first quadrant, the right-angled triangles P_1N_1O and $P'_1N'_1O$ are not only similar but also have their corresponding sides of the same sign. Hence, the trigonometrical ratios of the angle XOP_1, whether defined from the triangle P_1N_1O or from $P'_1N'_1O$ are the same in magnitude as well as in sign. Thus for any given angle, the trigonometrical ratios are unique.

Note. In case of a *positive acute angle* like XOP, we might take any point Q on OX as well, and draw QM perpendicular upon OP, and define $\sin XOP$ to be $\dfrac{\text{opposite side}}{\text{hypotenuse}}$ *i.e.*, $\dfrac{QM}{OQ}$, $\cos XOP$ to be $\dfrac{OM}{OQ}$ etc. Now the two triangles QOM and PON are easily seen to be similar and both have their sides all positive; so that $\dfrac{QM}{OQ} = \dfrac{PN}{OP}$, $\dfrac{OM}{OQ} = \dfrac{ON}{OP}$ etc. Hence the trigonometrical ratios of the angle XOP, even if defined from triangle QOM, will have the same values.

It may also be noted that *for angles of any magnitude, positive or negative,* any of the two arms may be supposed to be coincident with OX, and then the magnitude and sign of the angle will fix up the position of the other arm, and thereby will make the trigonometrical ratios unique.

10. Fundamental relations between the Trigonometrical ratios of any angle.

From the very definitions given in Art. 7 of the trigonometrical ratios of any angle XOP ($= \theta$) of whatever magnitude and sign, we at once derive the following relations :

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

and since $\sin \theta = \dfrac{PN}{OP}$, $\cos \theta = \dfrac{ON}{OP}$, $\tan \theta = \dfrac{PN}{ON}$, $\cot \theta = \dfrac{ON}{PN}$,

we get $\qquad \tan \theta = \dfrac{\sin \theta}{\cos \theta}$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}.$$

Again, since in the right-angled triangle OPN,
$$OP^2 = PN^2 + ON^2,$$
dividing by OP^2, ON^2 and PN^2 respectively, we get

$$\left(\frac{PN}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2 = 1 \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

$$\left(\frac{OP}{ON}\right)^2 = \left(\frac{PN}{ON}\right)^2 + 1 \qquad \cdots \qquad \cdots \qquad \text{(ii)}$$

$$\left(\frac{OP}{PN}\right)^2 = 1 + \left(\frac{ON}{PN}\right)^2. \qquad \cdots \qquad \cdots \qquad \text{(iii)}$$

From the definition of the trigonometrical ratios, (i) gives

$$(\sin \theta)^2 + (\cos \theta)^2 = 1.$$

Now, it is usual to write $(\sin \theta)^2$ in the form $\sin^2\theta$ and so for other ratios. The relation then reduces to the form

$$\sin^2\theta + \cos^2\theta = 1.$$

Similarly, (ii) and (iii) give respectively,

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

These formulæ are also used in the forms

$$\sin^2\theta = 1 - \cos^2\theta, \;\; \cos^2\theta = 1 - \sin^2\theta,$$

$$\sec^2\theta - \tan^2\theta = 1, \;\; \tan^2\theta = \sec^2\theta - 1, \text{ etc.}$$

Note. The fundamental formulæ derived in this article are very important, and are true for all values of θ whatever its magnitude and sign may be. For example, if we take $\frac{\theta}{2}$ in place of θ, we are simply taking a different angle for which the same relations are true, so that $\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$, etc.

11. Conversions of Trigonometrical ratios.

With the help of the formulæ of the previous article, we can express any trigonometrical ratio of an angle in terms of any other ratio for the same angle ; hence if the value of any trigonometrical ratio of an angle be given, we can find the value of any other ratio.

Ex. 1. *Express sin θ in terms of cot θ.*

From the formulæ $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta}$

and $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta,$

we get $\sin \theta = \dfrac{1}{\operatorname{cosec} \theta} = \dfrac{1}{\pm \sqrt{1 + \cot^2 \theta}}.$

Ex. 2. *Express cosec θ in terms of sec θ.*

$\operatorname{cosec} \theta = \pm \sqrt{1 + \cot^2 \theta} = \pm \sqrt{1 + \dfrac{1}{\tan^2 \theta}}$

$= \pm \sqrt{\dfrac{\tan^2 \theta + 1}{\tan^2 \theta}} = \pm \sqrt{\dfrac{\sec^2 \theta}{\sec^2 \theta - 1}} = \dfrac{\pm \sec \theta}{\sqrt{\sec^2 \theta - 1}}.$

Ex. 3. *If cos A = $\frac{12}{13}$, find tan A.*

We have $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\pm \sqrt{1 - \cos^2 A}}{\cos A}$

$= \dfrac{\pm \sqrt{1 - \frac{144}{169}}}{\frac{12}{13}} = \dfrac{\pm \frac{5}{13}}{\frac{12}{13}} = \pm \dfrac{5}{12}.$

A more practical method in such cases is however to construct a right-angled triangle with the numerator and denominator as the two suitable sides, as shown below.

Ex. 4. *If sec A = $\frac{41}{9}$, find cot A,*

Let APN be a triangle right-angled at N in which the hypotenuse $AP = 41$, $AN = 9$,

so that $\sec NAP = \dfrac{AP}{AN} = \dfrac{41}{9}.$

Thus $\angle NAP = A.$

Now, $PN^2 = AP^2 - AN^2 = 41^2 - 9^2$

$= 40^2,$

so that $PN = \pm 40.$

$\therefore \quad \cot A = \cot NAP = \dfrac{AN}{PN} = \pm \dfrac{9}{40}.$
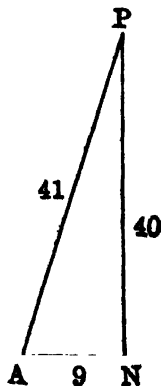
12. Restrictions on the magnitudes of Trigonometrical ratios.

From the relation $\sin^2\theta + \cos^2\theta = 1$, since $\sin^2\theta$ and $\cos^2\theta$ being square quantities are both positive, it is evident that neither $\sin^2\theta$ nor $\cos^2\theta$ can exceed 1, for if $\sin^2\theta$, for example, be greater than 1, $\cos^2\theta$ (which is a square quantity) becomes negative, which is impossible.   Thus, *sin θ as well as cos θ must have numerical values not exceeding* 1 ; in other words, both sin θ and cos θ must lie between + 1 and − 1 whatever the magnitude of θ may be.   Any value numerically greater than 1, like − 2 or + 3˙1 must be impossible for sin θ or cos θ so long θ is real.

sec θ and *cosec θ* therefore, being reciprocals of cos θ and sin θ respectively, *can never be numerically less than* 1.

tan θ and *cot θ* however, *can have any numerical value* greater than 1 or less than 1 according to the value of θ.

13. A few examples on the applications of the fundamental formulæ are given below.

Ex. 1. *Prove that* $\sqrt{\dfrac{1 + \cos\theta}{1 - \cos\theta}} = cosec\ \theta + cot\ \theta.$

[*C. U. 1937*]

$$\sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}} = \sqrt{\frac{(1 + \cos\theta)^2}{1 - \cos^2\theta}} = \sqrt{\frac{(1 + \cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{1 + \cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = cosec\ \theta + cot\ \theta.$$

Ex. 2. *Prove that*

$$\frac{1}{sec\ A + tan\ A} - \frac{1}{cos\ A} = \frac{1}{cos\ A} - \frac{1}{sec\ A - tan\ A}.$$

We have $\dfrac{1}{sec\ A + tan\ A} + \dfrac{1}{sec\ A - tan\ A}$

$$= \frac{sec\ A - tan\ A + sec\ A + tan\ A}{(sec\ A + tan\ A)(sec\ A - tan\ A)} = \frac{2\ sec\ A}{sec^2 A - tan^2 A}$$

$$= 2\ sec\ A = \frac{2}{\cos A} = \frac{1}{\cos A} + \frac{1}{\cos A}.$$

Hence, by transposition,

$$\frac{1}{\sec A + \tan A} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{\sec A - \tan A}.$$

Ex. 3.   *Prove that* $\dfrac{1 + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)}$
$$= \sin \theta \cos \theta \; (\sin \theta + \cos \theta).$$

We have $\dfrac{1 + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)(\cot \theta + \tan \theta)}$

$$= \frac{(\sin^2\theta + \cos^2\theta) + 2 \sin \theta \cos \theta}{(\sin \theta + \cos \theta)\left(\dfrac{\cos \theta}{\sin \theta} + \dfrac{\sin \theta}{\cos \theta}\right)}$$

$$= \frac{(\sin \theta + \cos \theta)^2}{(\sin \theta + \cos \theta)\left(\dfrac{\cos^2\theta + \sin^2\theta}{\sin \theta \cos \theta}\right)}$$

$$= \frac{(\sin \theta + \cos \theta) \sin \theta \cos \theta}{1}$$

$$= \sin \theta \cos \theta \; (\sin \theta + \cos \theta).$$

Ex. 4.   *If* $15 \sin^2\theta + 2 \cos \theta = 7$, *find* $\tan \theta$.

Here $15(1 - \cos^2\theta) + 2 \cos \theta = 7$,

whence $15 \cos^2\theta - 2 \cos \theta - 8 = 0$,

or, $(5 \cos \theta - 4)(3 \cos \theta + 2) = 0$, $\therefore$ $\cos \theta = \frac{4}{5}$, or, $-\frac{2}{3}$.

Case (i) when $\cos \theta = \frac{4}{5}$,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \tfrac{16}{25} = \tfrac{9}{25}.\quad \therefore \quad \sin \theta = \pm \tfrac{3}{5},$$

and so $\tan \theta = \dfrac{\sin \theta}{\cos \theta} = \pm \; \dfrac{3}{4}$.

Case (ii) when $\cos \theta = -\frac{2}{3}$,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \tfrac{4}{9} = \tfrac{5}{9}.\quad \therefore \quad \sin \theta = \pm \dfrac{\sqrt{5}}{3}.$$

$$\tan \theta = \dfrac{\sin \theta}{\cos \theta} = \pm \; \dfrac{\sqrt{5}}{2}$$

Examples II

Prove the following identities (*Ex. 1 to 24*) :-

1. $\dfrac{\sin A + \cos A}{\sec A + \operatorname{cosec} A} = \sin A \cos A$.

2. $\cot \theta + \tan \theta = \sec \theta \operatorname{cosec} \theta$.

3. $\dfrac{1}{1 + \tan A} = \dfrac{\cot A}{1 + \cot A}$.

4. $\operatorname{cosec}^6 A - \cot^6 A = 1 + 3 \operatorname{cosec}^2 A \cot^2 A$.

5. $\cos^6 A + \sin^6 A = 1 - 3 \sin^2 A \cos^2 A$.

6. $\dfrac{1}{\cos^2 A} - \dfrac{1}{\operatorname{cosec}^2 A - 1} = 1$.

7. $\cos A + \tan A \sin A = \sec A$.

8. $\sec^4 A + \tan^4 A = 1 + 2 \sec^2 A \tan^2 A$.

9. $\dfrac{1 + 3 \cos \theta - 4 \cos^3 \theta}{1 - \cos \theta} = (1 + 2 \cos \theta)^2$.

10. $(\cot \theta + \operatorname{cosec} \theta)^2 = \dfrac{1 + \cos \theta}{1 - \cos \theta}$.

11. $\dfrac{1 + \tan^2 \theta}{1 + \cot^2 \theta} = \left(\dfrac{1 - \tan \theta}{1 - \cot \theta}\right)^2$.

12. $\dfrac{\tan^2 a - \cot^2 a}{1 + \cot^2 a} = \dfrac{\sin^2 a - \cos^2 a}{\cos^2 a}$.

13. $1 + \tan \theta + \sec \theta = \dfrac{2}{1 + \cot \theta - \operatorname{cosec} \theta}$.

14. $\dfrac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \dfrac{1 + \sin \theta}{\cos \theta}$.

15. $\dfrac{\sin A - 2 \sin^3 A}{2 \cos^3 A - \cos A} = \tan A$.

16. $\sqrt{\dfrac{1 + \sin \theta}{1 - \sin \theta}} - \sec \theta = \sec \theta - \sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}}$

17. $\dfrac{\operatorname{cosec} A + \cot A}{\operatorname{cosec} A - \cot A} = \dfrac{\sin^2 A}{(1 - \cos A)^2}.$

18. $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A).$

19. $\dfrac{\sec \theta + \tan \theta}{\operatorname{cosec} \theta + \cot \theta} - \dfrac{\sec \theta - \tan \theta}{\operatorname{cosec} \theta - \cot \theta} = 2(\sec \theta - \operatorname{cosec} \theta).$

20. $\dfrac{1}{1 + \sin^2 \theta} + \dfrac{1}{1 + \operatorname{cosec}^2 \theta} = 1.$

21. $\dfrac{\sin^3 a + \cos^3 a}{\sin a + \cos a} + \dfrac{\sin^3 a - \cos^3 a}{\sin a - \cos a} = 2.$

22. $\dfrac{\tan \theta}{\sec \theta - 1} - \dfrac{\sin \theta}{1 + \cos \theta} = 2 \cot \theta.$

23. $\dfrac{\cos \theta + \cos \phi}{\sin \theta - \sin \phi} = \dfrac{\sin \theta + \sin \phi}{\cos \phi - \cos \theta}.$

24. $1 + 4 \operatorname{cosec}^2 \theta \cot^2 \theta = (\operatorname{cosec}^2 \theta + \cot^2 \theta)^2.$

25. Express $1 - 2 \sin \theta \cos \theta$ as a perfect square.

26. Express $2 \sec^2 \theta - \sec^4 \theta - 2 \operatorname{cosec}^2 \theta + \operatorname{cosec}^4 \theta$ in terms of $\tan \theta$.

27. Prove that
$$(\sin a \cos \beta + \cos a \sin \beta)(\sin a \cos \beta - \cos a \sin \beta)$$
$$= \sin^2 a - \sin^2 \beta.$$

28. If $\sin A + \sin^2 A = 1$, then $\cos^2 A + \cos^4 A = 1$.

29. (i) If $\sin \theta - \cos \theta = 0$, prove that $\sec \theta = \pm \sqrt{2}$.

(ii) If $7 \sin^2 \theta + 3 \cos^2 \theta = 4$, show that $\tan \theta \pm \dfrac{1}{\sqrt{3}}$.

(iii) If $3 \sin \theta + 4 \cos \theta = 5$, show that $\sin \theta = \tfrac{3}{5}$.

30. If $\tan \theta + \sec \theta = x$, show that $\sin \theta = \dfrac{x^2 - 1}{x^2 + 1}.$

31. If $\tan \theta = \dfrac{a}{b}$, find the value of $\dfrac{a \sin \theta - b \cos \theta}{a \sin \theta + b \cos \theta}.$

32.   If $1 + 4x^2 = 4x \sec A$, prove that
$$\sec A + \tan A = 2x \text{ or } 1/2x.$$

33.   Express $\sin a$ in terms of $\sec a$, and $\sec \theta$ in terms of $\cot \theta$.

34.   Given $\sin \theta = \frac{3}{5}$, $\cos \phi = \frac{12}{13}$, where θ and ϕ are acute angles, find the value of $\dfrac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$.

35.   If $\cos a + \sin a = \sqrt{2} \cos a$, prove that
$$\cos a - \sin a = \sqrt{2} \sin a.$$

36.   If $\tan A = \dfrac{1}{\sqrt{3}}$, find $\dfrac{\operatorname{cosec}^2 A - \sec^2 A}{\operatorname{cosec}^2 A + \sec^2 A}$.

37.   If $1 + \sin^2 A = 3 \sin A \cos A$, find $\tan A$.

38.   If $\tan \theta + \sin \theta = m$, $\tan \theta - \sin \theta = n$, prove that
$$m^2 - n^2 = 4 \sqrt{mn}.$$

39.   If $(a^2 - b^2) \sin \theta + 2ab \cos \theta = a^2 + b^2$, find $\tan \theta$ and $\operatorname{cosec} \theta$.

40.   If $\tan \theta = \dfrac{\sin a - \cos a}{\sin a + \cos a}$, prove that
$$\sqrt{2} \cos \theta = \sin a + \cos a.$$

41.   If $\cos^4 \theta + \cos^2 \theta = 1$, show that
  (i) $\tan^4 \theta + \tan^2 \theta = 1$.
  (ii) $\cot^4 \theta - \cot^2 \theta = 1$.

42.   If x and y are two unequal real quantities, show that the equations (i) $\sin^2 \theta = \dfrac{(x+y)^2}{4xy}$ and (ii) $\cos \theta = x + \dfrac{1}{x}$ are both impossible.

43.   Eliminate θ between
  (i) $x = a \cos \theta$, $y = b \sin \theta$.
  (ii) $x = c (\sec \theta + \tan \theta)$, $y = c (\sec \theta - \tan \theta)$.
  (iii) $a \cos \theta + b \sin \theta + c = 0$, $a' \cos \theta + b' \sin \theta + c' = 0$.
  (iv) $a \tan^2 \theta + b \tan \theta + c = a' \cot^2 \theta + b' \cot \theta + c' = 0$.

Examples II(A)

Prove the following identities (*Ex. 1 to 18*) :—

1. $\dfrac{\tan^3 a}{1 + \tan^2 a} + \dfrac{\cot^3 a}{1 + \cot^2 a} = \dfrac{1 - 2 \sin^2 a \cos^2 a}{\sin a \cos a}$.

2. $(\tan \theta + \cot \theta + \sec \theta)(\tan \theta + \cot \theta - \sec \theta) = \operatorname{cosec}^2 \theta$.

3. $\sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \operatorname{cosec} \theta$.
$$[\ C.\ U\ 1935\ ]$$

4. $(1 + \sin a - \cos a)^2 + (1 - \sin a + \cos a)^2$
$$= 4(1 - \sin a \cos a).$$

5. $\sin^6 a + \sin^4 a \cos^2 a - \sin^2 a \cos^4 a - \cos^6 a$
$$= \sin^2 a - \cos^2 a.$$

6. $3 (\sin \theta + \cos \theta) - 2 (\sin^3 \theta + \cos^3 \theta) = (\sin \theta + \cos \theta)^3$.

7. $\dfrac{1}{\operatorname{cosec} \theta - \cot \theta} - \dfrac{1}{\sin \theta} = \dfrac{1}{\sin \theta} - \dfrac{1}{\operatorname{cosec} \theta + \cot \theta}$.

8. $\dfrac{\cos x}{\sin x + \cos y} + \dfrac{\cos y}{\sin y - \cos x} = \dfrac{\cos x}{\sin x - \cos y} + \dfrac{\cos y}{\sin y + \cos x}$.

9. $(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 = \tan^2 \theta + \cot^2 \theta + 7$.

10. $(\sec \theta - \cos \theta)(\operatorname{cosec} \theta - \sin \theta)(\tan \theta + \cot \theta) = 1$.

11. $\dfrac{1 + (\operatorname{cosec} x \tan y)^2}{1 + (\operatorname{cosec} z \tan y)^2} = \dfrac{1 + (\cot x \sin y)^2}{1 + (\cot z \sin y)^2}$.

12. $\sec^3 a \operatorname{cosec}^3 a - 3 \sec a \operatorname{cosec} a = \tan^3 a + \cot^3 a$.

13. $\sin^6 A - \cos^6 A = (\sin A + \cos A)(\sin A - \cos A)$
$$\times (1 + \sin A \cos A)(1 - \sin A \cos A).$$

14. $\dfrac{\tan a}{(1 + \tan^2 a)^2} + \dfrac{\cot a}{(1 + \cot^2 a)^2} = \sin a \cos a$.

15. $\sin^2 \theta \tan \theta - \cos^2 \theta \cot \theta + \sec \theta \operatorname{cosec} \theta = 2 \tan \theta$.

16. $\dfrac{\cos^2 A - \sin^2 A}{\sin A \cos^2 A - \cos A \sin^2 A} = \operatorname{cosec} A + \sec A$.

17. $\dfrac{\tan^2 A + \cot^2 A}{\tan^2 A - \cot^2 A} = \dfrac{\sin^4 A + \cos^4 A}{\sin^2 A - \cos^2 A}$.

18. $(\sin \alpha \cos \beta - \cos \alpha \sin \beta)^2$
$+ (\cos \alpha \cos \beta + \sin \alpha \sin \beta)^2 = 1.$

19. If $\cos^2 A - \sin^2 A = \tan^2 B$,
then $\cos^2 B - \sin^2 B = \tan^2 A$.

20. If $\sin^4 x + \sin^2 x = 1$, the $\tan^4 x - \tan^2 x = 1$.

21. Show that the difference between $3 \sin^4 \theta - 2 \sin^6 \theta$ and $2 \cos^6 \theta - 3 \cos^4 \theta$ is the same for all values of θ.

22. If $x = \dfrac{1 + \sin \theta}{\cos \theta}$, show that $\dfrac{1}{x} = \dfrac{1 - \sin \theta}{\cos \theta}$.

23. If $\tan^2 A = 1 + 2 \tan^2 B$, show that $\cos^2 B = 2 \cos^2 A$.

24. If $\sin \alpha + \cos \alpha = 1$, the $\sin \alpha - \cos \alpha = \pm 1$.

25. If $a \cos \theta - b \sin \theta = c$, then show that
$a \sin \theta + b \cos \theta = \pm \sqrt{a^2 + b^2 - c^2}$.

26. If $(1 + \sin x)(1 + \sin y)(1 + \sin z)$
$= (1 - \sin x)(1 - \sin y)(1 - \sin z),$
prove that each is equal to $\pm \cos x \cos y \cos z$.

27. If $x \sin^3 \alpha + y \cos^3 \alpha = \sin \alpha \cos \alpha$, and
$x \sin \alpha - y \cos \alpha = 0$, then $x^2 + y^2 = 1$. [*C. U. 1937*]

28. If $\sin A = \dfrac{\sin x + \sin y}{1 + \sin x \sin y}$, show that
$\cos A = \pm \dfrac{\cos x \cos y}{1 + \sin x \sin y}$.

29. (i) If $\sin \alpha + \operatorname{cosec} \alpha = 2$,
then $\sin^n \alpha + \operatorname{cosec}^n \alpha = 2$.

(ii) If $\sec \alpha = \sec \beta \sec \gamma + \tan \beta \tan \gamma$,
then $\sec \beta = \sec \gamma \sec \alpha \pm \tan \gamma \tan \alpha$.

30. If $\dfrac{\cos^4 x}{\cos^2 y} + \dfrac{\sin^4 x}{\sin^2 y} = 1$, then $\dfrac{\cos^4 y}{\cos^2 x} + \dfrac{\sin^4 y}{\sin^2 x} = 1$.

TRIGONOMETRICAL RATIOS OF SOME STANDARD ANGLES

14. Ratios of 30°.
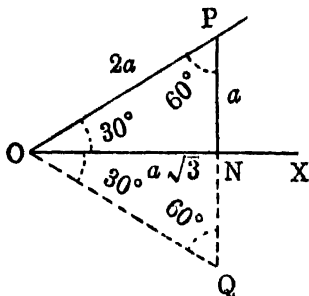
Let the angle XOP, which may be supposed to be traced out by a revolving line starting from OX, be 30°. Let PN be drawn perpendicular upon OX from any point P on OP. The angle OPN is then 60°.

Produce PN to Q, making $NQ = NP$. Join OQ. The triangles PON and QON are easily seen to be equal in all respects, and so $\angle OQN = \angle OPN = 60°$. Hence, the triangle OPQ is equilateral, and so $OP = PQ =$ double of PN.

Hence, in the above figure, if $PN = a$, then $OP = 2a$ and so $ON = \sqrt{OP^2 - PN^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$. The sides ON, PN and OP are all positive in this case, since the angle is acute.

Hence,

$$\sin 30° = \sin PON = \frac{PN}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30° = \frac{ON}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{PN}{ON} = \frac{1}{\sqrt{3}}$$

$$\cot 30° = \frac{ON}{PN} = \sqrt{3}$$

$$\operatorname{cosec} 30° = \frac{1}{\sin 30°} = 2$$

$$\sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}.$$

15.   Ratios of 45°.
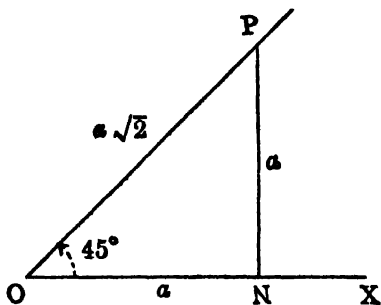
Let $\angle XOP = 45°$. PN is perpendicular on OX.   In the right-angled triangle PON, $\angle PON = 45°$.

Therefore, $\angle OPN$ is also $45°$ and so $ON = PN = a$ suppose. Then $OP = \sqrt{ON^2 + PN^2} = \sqrt{a^2 + a^2} = a\sqrt{2}$.

Hence,

$$\sin 45° = \frac{PN}{OP} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{ON}{OP} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{PN}{ON} = 1$$

$$\sec 45° = \operatorname{cosec} 45° = \sqrt{2}, \cot 45° = 1.$$

16. Ratios of 60°.
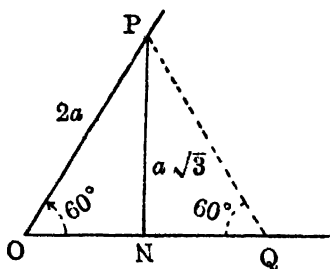
Let $\angle XOP = 60°$. Now PN being perpendicular upon OX, along NX cut off $NQ = ON$. Join PQ. Then the two triangles OPN and QPN are easily seen to be congruent. Hence, $\angle PQN = \angle PON = 60°$. Thus, the triangle POQ is equilateral, and so $OP = OQ =$ double of ON.

If $ON = a$, then $OP = 2a$ and hence $PN = \sqrt{OP^2 - ON^2} = a\sqrt{3}$.

Then
$$\sin 60° = \frac{PN}{OP} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{ON}{OP} = \frac{1}{2}$$

$$\tan 60° = \frac{PN}{ON} = \sqrt{3}$$

$$\cot 60° = \frac{1}{\sqrt{3}}, \ \sec 60° = 2, \ \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Note. It may be noted from the values of the ratios that $\sin 60° = \cos 30°$, $\cos 60° = \sin 30°$, $\tan 60° = \cot 30°$, $\cot 60° = \tan 30°$, $\sec 60° = \operatorname{cosec} 30°$, $\operatorname{cosec} 60° = \sec 30°$. It will be proved more generally, in the next chapter, that for any two complementary angles sine of one is the cosine of the other and *vice versa*, tangent of one is the co-tangent of the other, and secant of one is the cosecant of the other. The angle 45° being its own complement, therefore, it should have its sine and cosine equal to one another, as is actually seen to be the case.

17. Ratios of 90°.

Let XOP be an acute angle very nearly 90°. PN being
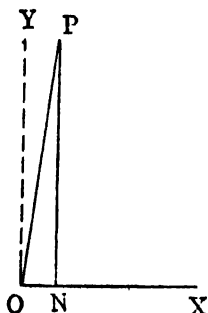
perpendicular upon OX, ON is extremely small, and as $\angle XOP$ approaches more and more to 90°, ON becomes smaller and smaller. The length OP may however remain finite, and PN and OP will approach each other more and more closely. Ultimately when $\angle XOP$ becomes 90°, OP and PN coincide, and ON becomes zero ultimately. Hence the ratio PN/OP becomes 1 and ON/OP becomes zero.

Thus, $\sin 90° = \dfrac{PN}{OP}$ in the limit $= 1$

$\cos 90° = \dfrac{ON}{OP}$ in the limit $= 0$

$\tan 90° = \dfrac{PN}{ON}$ in the limit $= \infty$* (infinity)

(since $ON \rightarrow 0$, whereas PN remains finite)

$\cot 90° = \dfrac{\cos 90°}{\sin 90°} = \dfrac{0}{1} = 0$

$\operatorname{cosec} 90° = 1$, $\sec 90° = OP/ON$ in the limit $=$

---

*The symbol ∞ is used to denote a quantity which exceeds any positive number, however large, and does not represent a definite number.

It should be noted that in determining $\tan 90°$, we may start with an angle XOP, slightly greater than 90°, (*i.e.*, in the second quadrant), and make it approach 90°. Then ON will be negative and $\rightarrow 0$, whereas PN is positive, Accordingly we may also write $\tan 90° = -\infty$. Thus strictly speaking, we should write $\tan 90° = \pm\infty$. Similar remarks apply for $\sec 90°$, $\cot 0°$, $\operatorname{cosec} 0°$.

18. Ratios of 0°.
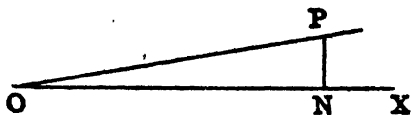
Let $\angle XOP$ be an infinitely small positive angle, and let PN be perpendicular on OX.

Then, PN is infinitely small, whereas OP is finite. Now, if $\angle XOP$ be taken less and less and ultimately becomes less than any quantity we can assign, we denote it by zero, and in this case PN practically vanishes, whereas OP and ON remaining finite, coincide. Hence, the ratio PN/OP becomes ultimately zero, and ON/OP becomes 1.

Hence, $\quad$ sin $0° = \dfrac{PN}{OP}$ in the limit $= 0$

$\qquad$ cos $0° = \dfrac{ON}{OP}$ in the limit $= 1$

$\qquad$ tan $0° = \dfrac{\sin 0°}{\cos 0°} = \dfrac{0}{1} = 0$

$\qquad$ cot $0° = \dfrac{ON}{PN}$ in the limit $= \infty$ *

$\qquad$ cosec $0° = \dfrac{OP}{PN}$ in the limit $= \infty$ *

$\qquad$ sec $0° = \dfrac{1}{\cos 0°} = \dfrac{1}{1} = 1.$

Note. Note that 0° and 90° being complementary,
$\quad$ sin 0° = cos 90°, cos 0° = sin 90° = 1, etc.

19. As the ratios of the standard angles 0°, 30°, 45°, 60°, and 90° are very often used, they should be remembered very

* See foot note of Art. 17.

8

carefully.   The first three ratios are given in the tabulated
form below.   The other three are reciprocals to these.

angle	sine	cosine	tangent
0° or 0°	0	1	0
30° or $\dfrac{\pi}{6}$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{\sqrt{3}}$
45° or $\dfrac{\pi}{4}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{1}{\sqrt{2}}$	1
60° or $\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{1}{2}$	$\sqrt{3}$
90° or $\dfrac{\pi}{2}$	1	0	$\pm \infty$

Note.   The following device may be of use in remembering the
sines and *cosines* of standard angles.   The sines of the angles 0°, 30°,
45°, 60°, 90° are respectively the square roots of the fractions

$$\tfrac{0}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{4}{4}$$

and cosines of these angles are the square roots from right to left.

20.   Examples worked out.

Ex. 1.   *If* $\theta = 30°$, *verify that* $\cos 2\theta = \dfrac{1 - tan^2\theta}{1 + tan^2\theta}$.

Here, $\cos 2\theta = \cos 60° = \dfrac{1}{2}$.   Also $\tan \theta = \tan 30° = \dfrac{1}{\sqrt{3}}$,

$$\therefore \quad \frac{1 - tan^2\theta}{1 + tan^2\theta} = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}.$$

Hence, $\cos 2\theta = \dfrac{1 - tan^2\theta}{1 + tan^2\theta}$.

Ex. 2. *Verify that*

$$\sin 30° = \sin 60° \cos 30° - \cos 60° \sin 30°.$$

The right-hand side, on substitution of the values,

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \sin 30°.$$

Hence the result.

Ex. 3. *Solve for* θ, *where* θ *is a positive acute angle, given* $\operatorname{cosec} \theta \cot \theta = 2\sqrt{3}$.

From the given equation, $\dfrac{1}{\sin \theta} \cdot \dfrac{\cos \theta}{\sin \theta} = 2\sqrt{3}$,

or, $\cos \theta = 2\sqrt{3} \sin^2 \theta = 2\sqrt{3}(1 - \cos^2 \theta)$,

whence, $2\sqrt{3} \cos^2 \theta + \cos \theta - 2\sqrt{3} = 0$

giving $\cos \theta = \dfrac{-1 \pm \sqrt{1 + 48}}{4\sqrt{3}} = \dfrac{-1 \pm 7}{4\sqrt{3}}$.

Since θ is a positive acute angle, $\cos \theta$ is positive, and so rejecting the negative value,

$$\cos \theta = \frac{6}{4\sqrt{3}} = \frac{\sqrt{3}}{2} = \cos 30°. \quad \therefore \quad \theta = 30° \quad i.e., \quad \frac{\pi}{6}.$$

Examples III

Verify the results (*Ex. 1 to 6*) :

1. $1 - 2 \sin^2 30° = 2 \cos^2 30° - 1 = \cos 60°.$

2. $\dfrac{2 \tan 30°}{1 - \tan^2 30°} = \sqrt{3}.$

3. $\cos \dfrac{\pi}{3} \cos \dfrac{\pi}{4} + \sin \dfrac{\pi}{3} \sin \dfrac{\pi}{4} = \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{6} + \cos \dfrac{\pi}{4} \sin \dfrac{\pi}{6}.$

4. (i) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}.$

    (ii) $\cos A = \cos^2 B - \sin^2 B$, where $A = 60°$, $B = 30°$.

5. $\sin 3A = 3 \sin A - 4 \sin^3 A$, where $A = \dfrac{\pi}{6}$.

6. $\operatorname{cosec}^2 45°. \sec^2 30° (\sin^3 30° + 4 \cot^2 45° - \sec^2 60°) = \frac{1}{3}$.

7. If $\tan^2 \dfrac{\pi}{4} - \cos^2 \dfrac{\pi}{3} = x \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{4} \tan \dfrac{\pi}{3}$, find x.

8. If θ be a positive acute angle, find θ, when

(i) $2 \sin^2 \theta = 3 \cos \theta$.

(ii) $\tan \theta + \cot \theta = 2$.

(iii) $\operatorname{cosec}^2 \theta + 5 = 3 \sqrt{3} \cot \theta$.

(iv) $\sin \theta + \cos \theta = \sqrt{2}$.

(v) $2 (\cos^2 \theta - \sin^2 \theta) = 1$.

(vi) $6 \sin^2 \theta - 11 \sin \theta + 4 = 0$.

(vii) $\dfrac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta} = \dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$.

9. Given θ and ϕ to be positive acute angles, and $\tan (\theta + \phi) = \sqrt{3}$, $\tan (\theta - \phi) = 1$, determine θ and ϕ.

10. Find α and β (α and β being positive acute angles), if

$$\sin (2\alpha - \beta) = 1,$$

and $\cos (\alpha + \beta) = \frac{1}{2}$.

11. Find A, B, C (A, B, C being positive acute angles), if

$$\sin (B + C - A) = 1,$$
$$\cos (C + A - B) = 1,$$

and $\tan (A + B - C) = 1$.

12. Find the numerical values of :—

(i) $\cot^2 \dfrac{\pi}{6} - 2 \cos^2 \dfrac{\pi}{3} - \dfrac{3}{4} \sec^2 \dfrac{\pi}{4} - 4 \sec^2 \dfrac{\pi}{6}$.

(ii) $3 \tan^2 45° - \sin^2 60° - \frac{1}{2} \cot^2 30° + \frac{1}{3} \sec^2 45°$.

TRIGONOMETRICAL RATIOS OF ANGLES ASSOCIATED WITH A GIVEN ANGLE θ

21.   Ratios of the angle $(-\theta)$ *in terms of those of* θ, *θ having any magnitude.*
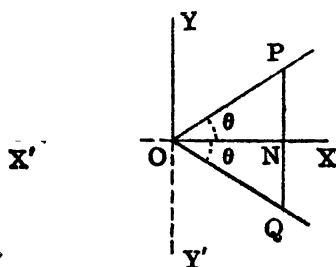
Fig. (i)
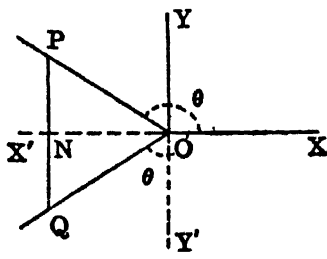
Fig. (ii)
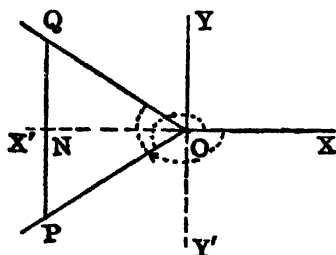
Fig. (iii)
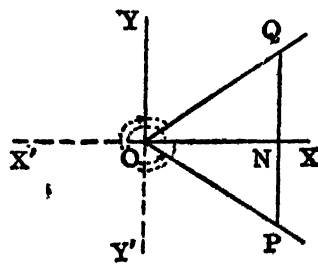
Fig. (iv)

Let the $\angle XOP$ be θ and the $\angle XOQ$ described clock-wisely be $-\theta$. From any point P on OP draw PN perpendicular to OX [or OX' as in Figs. (ii) and (iii)], and produce it to meet OQ at Q say.

Now, $\angle XOP$ (measured anti-clockwisely) being equal to $\angle XOQ$ (measured clockwisely), $\angle PON = \angle QON$ in magnitude in all the figures, and therefore, the two rt.-angled triangles PON and QON are congruent. The corresponding sides are therefore equal in magnitude. Considering the signs of these sides according to the usual convention, we get in all the figures,

$$QN = - PN, \text{ and } OQ = OP.$$

(both OP and QQ being always considered positive)

Hence, from definition,

$$\sin(-\theta) = \frac{QN}{OQ} = \frac{-PN}{OP} = -\sin\theta$$

$$\cos(-\theta) = \frac{ON}{OQ} = \frac{ON}{OP} = \cos\theta$$

$$\tan(-\theta) = \frac{QN}{ON} = \frac{-PN}{ON} = -\tan\theta$$

and the reciprocals of these give,

$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta,$$

$$\sec(-\theta) = \sec\theta,$$

$$\cot(-\theta) = -\cot\theta.$$

22.   Ratios of $(90° - \theta)$.

Let the $\angle XOP$ traced out by a revolving line be θ, and let another revolving line, starting from OX trace out the angle $XOY = 90°$ and then revolve back, tracing out $\angle YOQ = \theta$ in the clockwise direction, so that $\angle XOQ = 90° - \theta$.

Take two equal lengths OP and OQ along OP and OQ respectively and draw PN and QM perpendiculars on OX.

If *OP be in the first or third quadrant* as in Fig. (i) and Fig. (iii), OQ also lies in the same quadrant. If OP lies in the second quadrant as in Fig. (ii), OQ lies in the fourth quadrant ; and if OP lies in the fourth, OQ lies in the

second, as in Fig. (iv). Now, $\angle XOP$ being equal to $\angle YOQ$ in magnitude, $\angle PON = \angle OQM$, and since $OP = OQ$,
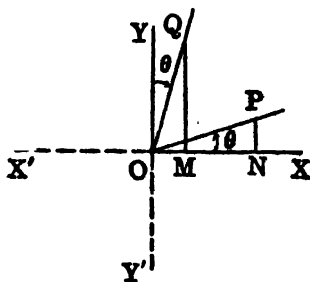
Fig. (i)
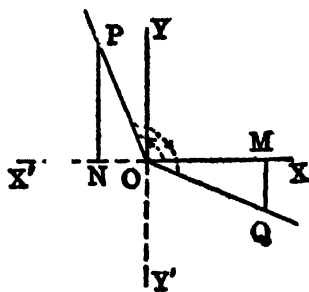
Fig. (ii)
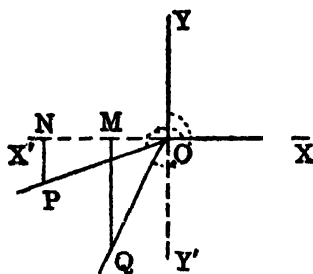
Fig. (iii)
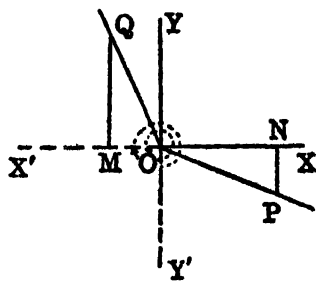
Fig. (iv)

the two right-angled triangles PON, OQM are congruent. The corresponding sides are therefore equal in magnitude. Considering signs as well, we get in all the figures,

$$QM = ON, \quad OM = PN, \quad OQ = OP.$$

Hence, from definition,

$$\sin(90° - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos\theta$$

$$\cos(90° - \theta) = \frac{OM}{OQ} = \frac{PN}{OP} = \sin\theta$$

$$\tan(90° - \theta) = \frac{QM}{OM} = \frac{ON}{PN} = \cot\theta.$$

The reciprocals of these are

$$\operatorname{cosec} (90° - \theta) = \sec \theta,$$
$$\sec (90° - \theta) = \operatorname{cosec} \theta,$$
$$\cot (90° - \theta) = \tan \theta.$$

Obs. The angles $(90° - \theta)$ is the complement of θ, and we derive the result that for a pair of complementary angles, since of one is the cosine of the other and *vice versa*, tangent of one is the cotangent of the other and secant of one is the cosecant of the other. This was verified in the last chapter in connection with the complementary pairs 30° and 60°, as also 0° and 90°.

23. Ratios of $(90° + \theta)$.
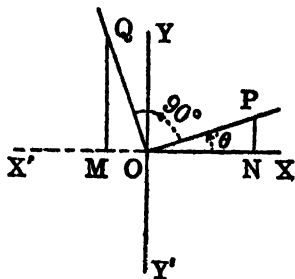
Fig. (i)
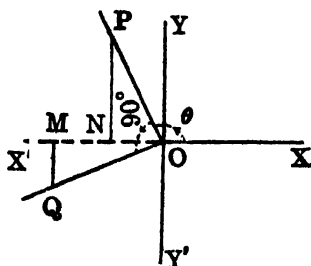
Fig. (ii)
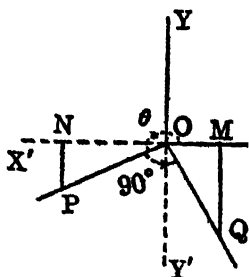
Fig. (iii)
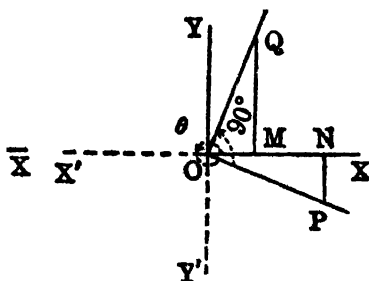
Fig. (iv)

Let a revolving line, starting from OX, trace out an $\angle XOP = \theta$, and further trace out an $\angle POQ = 90°$, so that $\angle XOQ = 90° + \theta$.

Cut off $OP = OQ$ along OP and OQ respectively and let PN, QM be perpendiculars on OX (produced where necessary).

Now, OQ being perpendicular to OP, the $\angle PON =$ the complement of $\angle QOM = \angle OQM$ in magnitude, and since $OP = OQ$, the two right-angled triangles OPN and OQM are congruent. The corresponding sides are therefore equal. Considering signs as well, we get, for all the figures,

$$QM = ON, \ OM = -PN, \ OQ = OP.$$

Hence, from definition,

$$\sin (90° + \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{ON}{OP} = \cos \theta$$

$$\cos (90° + \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin \theta$$

$$\tan (90° + \theta) = \frac{QM}{OM} = \frac{ON}{-PN} = -\cot \theta$$

and considering their reciprocals,

$$\operatorname{cosec} (90° + \theta) = \sec \theta,$$
$$\sec (90° + \theta) = -\operatorname{cosec} \theta,$$
$$\cot (90° + \theta) = -\tan \theta.$$

24. Ratios of $(180° - \theta)$.
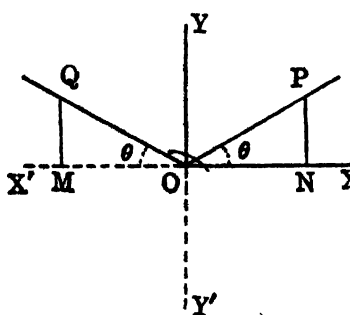
Fig. (i)   Fig. (ii)
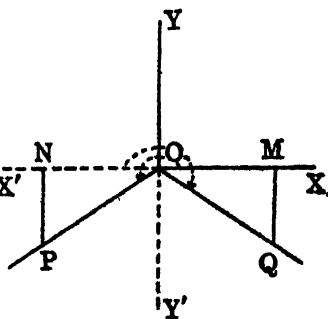
Let $\angle XOP = \theta$ be traced out by a revolving line, and let another revolving line, starting from OX, trace out an angle $180°$ coming up to OX' and then revolve back and describe an angle $X'OQ = \theta$, so that $\angle XOQ = 180° - \theta$.

Two figures are given here, one with OP in the first quadrant and another with OP in the third quadrant. The two other figures may easily be drawn by the students.

Now cut off $OP = OQ$, and draw PN and QM perpendiculars on OX (or OX' as the case may be). Then $\angle PON = \angle QOM$ in magnitude, and $OP = OQ$. Hence, the right-angled triangles PON and QOM are congruent, and so have their corresponding sides equal in magnitude. Taking into consideration the signs, we get for all the figures,

$$QM = PN, \quad OM = -ON, \quad OQ = OP.$$

Hence, for all values of θ,

$$\sin(180° - \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{PN}{OP} = \sin\theta$$

$$\cos(180° - \theta) = \frac{OM}{OP} = \frac{-ON}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{QM}{OM} = \frac{PN}{-ON} = -\tan\theta$$

and so taking reciprocals,

$$\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta,$$

$$\sec(180° - \theta) = -\sec\theta,$$

$$\cot(180° - \theta) = -\cot\theta.$$

Note. The first two formulæ may be expressed in the form *"sines of supplementary angles are equal, and cosines of supplementary angles are equal in magnitude but opposite in sign"*.

25. Ratios of $(180° + \theta)$.

Let a revolving line starting from OX, trace out an angle $XOP = \theta$, and further trace out an angle $POQ = 180°$, so that $\angle XOQ = 180° + \theta$.

OP and *OQ* are then in one straight line.
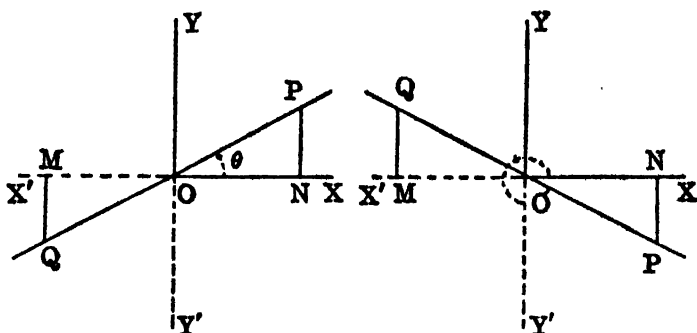
Fig. (i)   Fig. (ii)

Cut off *OP = OQ*, and draw *PN* and *QM* perpendiculars on *XOX'*.

Two figures are given here with *OP* in the first and fourth quadrants, and the other two may be similarly drawn.

Now, *POQ* being a straight line in this case, $\angle PON = \angle QOM$ in magnitude. Also, *OP = OQ*. Hence, the right-angled triangles *PON* and *QOM* are congruent and have their corresponding sides equal in magnitude. Considering signs, we get in all cases,

$$QM = -PN, \quad OM = -ON, \quad OQ = OP.$$

Thus, for all values of θ,

$$\sin(180° + \theta) = \sin XOQ = \frac{QM}{OQ} = \frac{-PN}{OP} = -\sin\theta$$

$$\cos(180° + \theta) = \frac{OM}{OQ} = \frac{-ON}{OP} = -\cos\theta$$

$$\tan(180° + \theta) = \frac{QM}{OM} = \frac{-PN}{-ON} = \frac{PN}{ON} = \tan\theta$$

and so,

$$\csc (180° + \theta) = - \csc \theta,$$
$$\sec (180° + \theta) = - \sec \theta,$$
$$\cot (180° + \theta) = \cot \theta.$$

26. Ratios of $(270° - \theta)$.
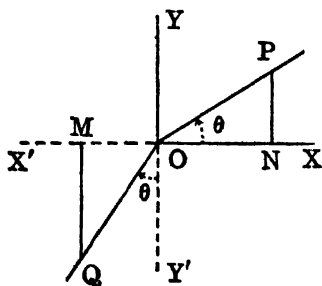
Fig. (i)   Fig. (ii)
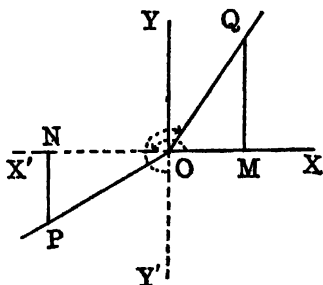
Let $\angle XOP = \theta$ be traced out by a revolving line, and let another revolving line trace out an angle $XOY' = 270°$, thereby coming up to the position OY', and then revolve back, tracing out an angle $Y'OQ = \theta$, so that $\angle XOQ = 270° - \theta$.

Two figures are given here with OP in the first and third quadrants. The other two may be drawn similarly.

Cut off $OP = OQ$ and draw PN, QM perpendiculars on XOX'.

Since $\angle XOP = \angle Y'OQ$ in magnitude, we easily derive that $\angle PON = \angle OQM$ in magnitude. Also $OP = OQ$. Hence, the two right-angled triangles OPN and OQM are congruent. Considering signs, we get for all the figures,

$$QM = - ON, \quad OM = - PN, \quad OQ = OP.$$

Hence, for all values of θ,

$$\sin (270^\circ - \theta) = \sin \angle XOQ = \frac{QM}{OQ} = \frac{-ON}{OP} = -\cos \theta$$

$$\cos (270^\circ - \theta) = \frac{OM}{OQ} = \frac{-PN}{OP} = -\sin \theta \; ;$$

$$\tan (270^\circ - \theta) = \frac{QM}{OM} = \frac{-ON}{-PN} = \frac{ON}{PN} = \cot \theta \; ;$$

and thus,

$$\csc (270^\circ - \theta) = -\sec \theta,$$

$$\sec (270^\circ - \theta) = -\csc \theta,$$

$$\cot (270^\circ - \theta) = \tan \theta.$$

27. Ratios of $(270^\circ + \theta)$.

We may proceed geometrically as in the previous cases. Otherwise we may proceed as follows :

$\sin (270^\circ + \theta) = \sin (180^\circ + 90^\circ + \theta) = -\sin (90^\circ + \theta)$ [from § 25

$\qquad\qquad = -\cos \theta \qquad \cdots \qquad\qquad \cdots \qquad$ [from § 23

$\cos (270^\circ + \theta) = \cos (180^\circ + 90^\circ + \theta) = -\cos (90^\circ + \theta)$

$\qquad\qquad = -(-\sin \theta) = \sin \theta$

$\tan (270^\circ + \theta) = \dfrac{\sin (270^\circ + \theta)}{\cos (270^\circ + \theta)} = \dfrac{-\cos \theta}{\sin \theta} = -\cot \theta \; ;$

and hence,

$$\csc (270^\circ + \theta) = -\sec \theta,$$

$$\sec (270^\circ + \theta) = \csc \theta,$$

$$\cot (270^\circ + \theta) = -\tan \theta.$$

Note. The ratios of $180^\circ - \theta$, $180^\circ + \theta$, $270^\circ - \theta$ can be similarly deduced from the formulæ for ratios of $90^\circ \pm \theta$.

28. Ratios of $(360^\circ - \theta)$, $(360^\circ + \theta)$ and $(n. 360^\circ \pm \theta)$.

It has already been remarked in Art. 2, Chapter I, that angles which differ by complete multiples of 360°, i.e., by an exact number of complete revolutions, have the final positions of the revolving lines coincident, if the initial lines

are the same. Hence, all the trigonometrical ratios of two such angles must be identical in magnitude as well as in sign.

Thus, trigonometrical ratios of $360° - \theta$ must be same as those of $-\theta$. Hence,

$$\sin (360° - \theta) = \sin (-\theta) = -\sin \theta$$
$$\cos (360° - \theta) = \cos (-\theta) = \cos \theta$$
$$\tan (360° - \theta) = \tan (-\theta) = -\tan \theta, \text{ etc.}$$

Trigonometrical ratios of $360° + \theta$, or of $360° \times n \pm \theta$, where n is an integer, positive or negative, must similarly be same as those of θ, or of $\pm \theta$.

Thus, *in determining trigonometrical ratios of angles, complete multiples of* $360°$ *(i.e.,* 2π*) may be always added or subtracted.*

29. All the above results may, for easy remembrance, be summed up in a **simple rule.**

If θ be associated with an even multiple of 90° by + or − sign, (*e.g.*, $180° - \theta$, $180° + \theta$, $360° - \theta$, $360° + \theta$, etc.) the ratio is not altered in form (*i.e.*, sine remains sine, cosine remains cosine, etc.).   To determine the sign, assuming θ to be acute, find out the quadrant in which the associated angle lies, and determine the sign according to the rule, "all, sin, tan, cos".

If θ be associated with an odd multiple of 90° by + or − sign, (*e.g.*, $90° - \theta$, $90° + \theta$, $270° - \theta$, $270° + \theta$, etc.) the ratio is altered (sine becomes cosine, cosine becomes sine, tangent becomes cotangent, etc.).   Moreover, the sign of the result is determined as in the previous paragraph.

Example. *Consider formulæ for tan* $(270° - \theta)$ *and sec* $(180° + \theta)$.

$270° - \theta = 3.90° - \theta$ (multiple of $90°$ is odd).

Hence, the ratio will be altered, tan changing into cot. Moreover, θ being assumed acute (whether it is actually so or not, it does not matter), $270° - \theta$ falls in the third quadrant, where tan is positive.

Hence, $\tan (270° - \theta) = + \cot \theta$.

$180° + \theta$ has got θ associated with even multiple of $90°$. Hence, the ratio does not alter in form, sec remaining sec. Also, $180° + \theta$ falls in the third quadrant, if θ be assumed acute, where sec (by the rule "all, sin, tan, cos") is negative.

Hence, $\sec (180° + \theta) = - \sec \theta$.

N. B. The angle '$-\theta$' may be written as $0°360° - \theta$, and 0 may be considered *even* in applying the above rule.

Thus, θ being supposed acute, $-\theta$ falls in the fourth quadrant, where cos and sec only are positive. The form of the ratio not changing in this case, $\sin (-\theta) = -\sin \theta$, $\cos (-\theta) = +\cos \theta$, etc.

30. Special angles *(outside the first quadrant)*.

In Art. 24, putting $\theta = 60°$, $45°$, $30°$ and $0°$ respectively we can deduce the following results :

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2} ; \qquad \cos 120° = -\cos 60° = -\frac{1}{2} .$$

$$\sin 135° = \sin 45° = \frac{1}{\sqrt{2}} ; \qquad \cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}.$$

$$\sin 150° = \sin 30° = \frac{1}{2} ; \qquad \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}.$$

$$\sin 180° = \sin 0° = 0 ; \qquad \cos 180° = -\cos 0° = -1.$$

And similarly from Arts. 27 and 28, putting $\theta = 0$,

$$\sin 270° = -\cos 0° = -1 ; \qquad \cos 270° = \sin 0° = 0 ;$$

$$\sin 360° = \sin 0° = 0 ; \qquad \cos 360° = \cos 0° = 1.$$

From the above we get,

$$\tan 180° = 0 ; \quad \tan 270° = \pm \infty ; \quad \tan 360° = 0.$$

Examples worked out.

Ex. 1. *Find the value of cot* $(-1575°)$.

$$\cot \ (-1575°) = -\cot \ (1575°) = -\cot \ (4 \times 360° + 185°)$$
$$= -\cot \ (135°) = -\cot \ (180° - 45°)$$
$$= \cot 45° = 1.$$

Ex. 2. *Find the value of cot* $\theta - \tan \theta$, *where* $\theta = \dfrac{17\pi}{3}$.

$\dfrac{17\pi}{3} = 6\pi - \dfrac{\pi}{3}$, and omitting complete multiples of $360°$ *i.e.*, of 2π, whereby trigonometrical ratios are not altered, we get,

$$\cot \frac{17\pi}{3} = \cot\left(-\frac{\pi}{3}\right) = -\cot \frac{\pi}{3} = -\cot 60° = -\frac{1}{\sqrt{3}}.$$
$$\tan \frac{17\pi}{3} = \tan\left(-\frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\tan 60° = -\sqrt{3}.$$
$$\therefore \quad \cot \theta - \tan \theta = -\frac{1}{\sqrt{3}} + \sqrt{3} = \frac{3-1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Ex. 3. *Prove that*

$$\sin \ (420°) \ \cos \ (390°) + \cos \ (-300°) \ \sin \ (-330°) = 1.$$
$$[\ H. \ S. \ 1962 \ ]$$

L. H. side $= \sin \ (360° + 60°) \ \cos \ (360° + 30°)$
$$+ \cos \ (-360° + 60°) \ \sin \ (-360° + 30°)$$
$$= \sin 60° \ \cos 30° + \cos 60° \ \sin 30°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1.$$

Ex. 4. *Express cot* $(-1358°)$ *in terms of the ratio of a positive angle less than* $45°$.

$$\cot \ (-1358°) = \cot \ (-4 \times 360° + 82°)$$
$$= \cot 82° = \cot \ (90° - 8°)$$
$$= \tan 8°.$$

Note. Ratios of angles of any magnitude and sign can always be expressed in terms of a ratio of a positive angle less than $45°$.

Ex. 5. *Express*

$$\frac{cos\ (90° + \theta)\ sec\ (-\theta)\ tan\ (180° - \theta)}{sec\ (360° + \theta)\ sin\ (180° + \theta)\ cot\ (90° - \theta)}\ in\ its\ simplest$$

form.

The given expression

$$= \frac{- \sin \theta.\sec \theta.(- \tan \theta)}{\sec \theta.(- \sin \theta).\tan \theta}$$

$$= - 1.$$

Examples IV

1. Write down the values of $\sin 150°$, $\cot 840°$, $\csc (- 660°)$ and $\tan (- 1125°)$.

2. Find the values of $\sin \left(- \frac{11\pi}{4} \right)$, $\csc \left(\frac{16\pi}{3} \right)$, $\tan \left(\frac{3\pi}{2} + \frac{\pi}{3} \right)$ and $\cos \left(\frac{5\pi}{2} - \frac{19\pi}{3} \right)$.

3. Evaluate $\sin \left(- 1230° \right) - \cos \left\{ \left(2n + 1 \right)\pi + \frac{\pi}{3} \right\}$, where n is a negative integer.

4. Find the value of $\sin \left\{ n\pi + (- 1)^n \frac{\pi}{3} \right\}$, where n is any integer.

5. Find all the values of

(i) $\tan \left\{ \frac{n\pi}{2} + (- 1)^n \frac{\pi}{4} \right\}$;

(ii) $\csc \left\{ \frac{n\pi}{2} + (- 1)^n \frac{\pi}{6} \right\}$,

where n is any integer.

6. Show that $\cos \left(2m\pi \pm \frac{\pi}{3} \right)$ and $\tan \left(m\pi + \frac{\pi}{6} \right)$ have one value each for all integral values of m.

4

7. Prove that, n being any integer

(i) $\cos (n\pi + a) = (-1)^n \cos a$.

(ii) $\tan (n\pi - a) = -\tan a$.

8. Prove that

(i) $\cos \theta = -\cos (\theta - 180°)$.

(ii) $\tan \theta = -\cot (\theta - \frac{3}{2}\pi)$.

9. Prove that

(i) $\sin (780°) \cos (390°) - \sin (330°) \cos (-300°) = 1$.

(ii) $\cos 306° + \cos 234° + \cos 162° + \cos 18° = 0$.

(iii) $\dfrac{\sin 250° + \tan 290°}{\cot 200° + \cos 340°} = -1$.

10. Simplify

$$\frac{\sin^3(\pi + \theta)}{\cos^2(\frac{1}{2}\pi + \theta)} \cdot \frac{\tan (2\pi - \theta)}{\operatorname{cosec}^2\theta} \cdot \frac{\sec^2(\pi - \theta)}{\sin (\pi - \theta)}$$

and determine its value when $\theta = 225°$.

11. Prove that

$$\sin (\tfrac{1}{2}\pi + \theta) \cos (\pi - \theta) \cot (\tfrac{3}{2}\pi + \theta)$$
$$= \sin (\tfrac{1}{2}\pi - \theta) \sin (\tfrac{3}{2}\pi - \theta) \cot (\tfrac{1}{2}\pi + \theta).$$

12. Evaluate

(i) $\sin^2 \dfrac{\pi}{4} + \sin^2 \dfrac{3\pi}{4} + \sin^2 \dfrac{5\pi}{4} + \sin^2 \dfrac{7\pi}{4}$.

(ii) $\cot \dfrac{\pi}{20} \cot \dfrac{3\pi}{20} \cot \dfrac{5\pi}{20} \cot \dfrac{7\pi}{20} \cot \dfrac{9\pi}{20}$.

(iii) $\sin x + \sin (\pi + x) + \sin (2\pi + x) + \cdots\cdots$ to n terms.

13. If $\tan \theta = \frac{5}{12}$ and $\cos \theta$ is negative, find the value of

$$\frac{\sin \theta + \cos (-\theta)}{\sec (-\theta) + \tan \theta}.$$

14. An angle θ lies between 180° and 270°, and $\operatorname{cosec} \theta = -\frac{7}{3}$. Find $\cot \theta$.

15. Express in terms of ratios of positive angles less than $45°$:

    (i) $\cot(-1054°)$.   (ii) $\sin(1145°)$.

    (iii) $\sec(-1491°)$.   (iv) $\cos\dfrac{35\pi}{9}$.

16. Find the values of θ, when

    (i) $\tan\theta = -\sqrt{3}$ and θ lies between $270°$ and $360°$.

    (ii) $\cos\theta = -\frac{1}{2}$, and $450° < \theta < 540°$.

17. Solve for θ, giving all the possible values, when $0° < \theta < 360°$:

    (i) $\cos\theta + \sqrt{3}\sin\theta = 2$.   [*C. U. 1936*]

    (ii) $2\sin^2\theta + 3\cos\theta = 0$.

    (iii) $3(\sec^2\theta + \tan^2\theta) = 5$.

    (iv) $\cot\theta + \tan\theta = 2\sec\theta$.

    (v) $1 - 2\sin\theta - 2\cos\theta + \cot\theta = 0$.

18. If A, B, C be the angles of a triangle, show that
$$\sin(A+B) - \cos C = \cos(A+B) + \sin C.$$

19. If A, B, C be the angles of a triangle, show that
$$\frac{\tan(B+C) + \tan(C+A) + \tan(A+B)}{\tan(\pi-A) + \tan(2\pi-B) + \tan(3\pi-C)} = 1.$$

20. If A, B, C, D be the angles of a quadrilateral, show that
$$\cos\tfrac{1}{2}(A+C) + \cos\tfrac{1}{2}(B+D) = 0.$$

If the quadrilateral be cyclic, then
$$\cos A + \cos B + \cos C + \cos D = 0.$$

SIMPLE PRACTICAL APPLICATIONS OF TRIGONOMETRY

(*Heights and Distances*)

31. One of the most important applications of Trigonometry is in the determination of *heights and distances* of distant objects which are not directly measurable, by observations of angles subtended by those objects at the eye of the observer. These angles may be measured by instruments known as Sextants or Theodolites or by other angle-measuring instruments. Thus, Trigonometry plays a very important part in *land survey*. It is also extensively used by Astronomers in determining the distances of the heavenly bodies like the sun, the moon and the stars.

Two angles are very often used in the practical applications of Trigonometry, and they are defined as follows :
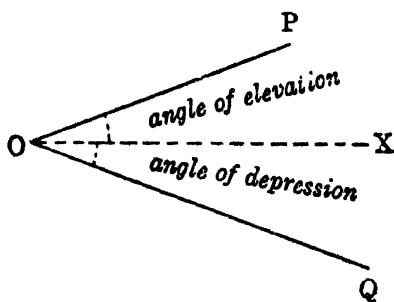
If a horizontal line *OX* be drawn through *O*, the eye of an observer, the angle which the line joining *O* to a point *P* above *OX* makes with *OX* is called the **Angle of Elevation** or *altitude of P* as seen from *O*.

If Q be below the horizontal line OX, the angle XOQ measured below OX is called the **Angle of Depression** of Q as seen from O.

32. Illustrative Examples.

Ex. 1. *From a distance of 40 feet from the foot of a palm tree in a horizontal field, the angle of elevation of the top of the tree is observed to be 60°. Find the height of the tree.*
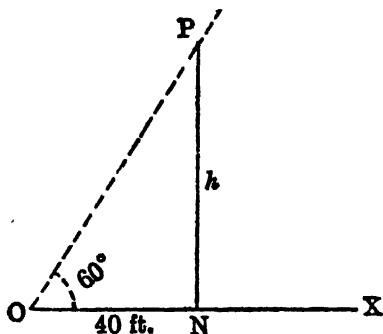
Let h ft. be the height of the tree PN, and $\angle NOP$, the angle of elevation of P as seen from O, where $ON = 40$ ft., is 60°.

Then, $\dfrac{h}{40} \cdot\cdot \tan PON = \tan 60° = \sqrt{3}$;

$\therefore \qquad h = 40 \sqrt{3}$ ft. $= 69\cdot28\cdots\cdots$ ft.

Ex. 2. *From one bank of a river, the top of a building just on the opposite bank is observed to have an elevation of 45°. On receding 50 ft. from the bank, perpendicular to its edge, the angle of elevation becomes 30°. Find the breadth of the river and the height of the building.*

AQ being the breadth of the river, PQ the height of the building, $\angle PAQ = 45°$. Also, AB being 50 ft. $\angle PBQ = 30°$.
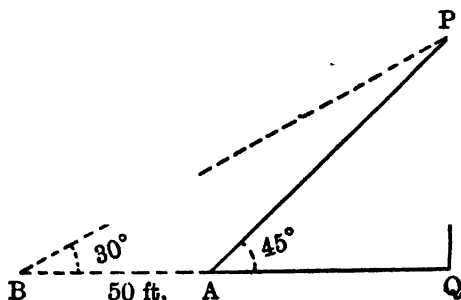
Now, $\dfrac{BQ}{PQ} = \cot 30°$, $\dfrac{AQ}{PQ} = \cot 45°$.

Hence, subtracting, $\dfrac{AB}{PQ} = \cot 30° - \cot 45°$,

or, $\dfrac{50}{PQ} = \sqrt{3} - 1$;

$\therefore \quad PQ = \dfrac{50}{\sqrt{3}-1} = \dfrac{50(\sqrt{3}+1)}{2} = 68\!\cdot\!3$ ft. nearly.

Also, $\dfrac{AQ}{PQ} = \cot 45° = 1$; $\therefore AQ = PQ = 68\!\cdot\!3$ ft.

Thus, the breadth of the river and the height of the building are both 68·3 ft. nearly.

Ex. 3. *The angles of depression and elevation of the top of a tower* 50 *ft. high from the top and bottom of a second tower are* 60° *and* 20° *respectively. Find the height of the second tower to the nearest foot.* [*Given* $\cot 20° = 2\!\cdot\!747.$]

PQ is the second tower, and $\angle XPA = 60°$, $\angle BQA = 20°$, $AB = 50$ ft., AC is parallel to BQ or PX, so that $\angle PAC$ = the alternate angle $XPA = 60°$.

Now, $\dfrac{QB}{AB} = \cot 20°$,   ∴ $QB = AB \cot 20°$.

Also, $\dfrac{PC}{CA} = \tan PAC = \tan 60°$;

. $PC = CA \tan 60° = QB \tan 60°$

$= AB \cot 20° \tan 60°$.
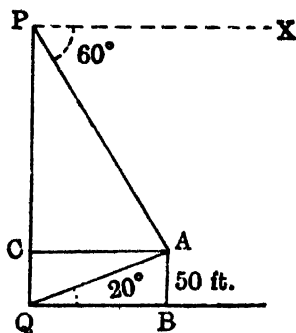
∴   height $PQ = PC + CQ = PC + AB$

$= AB (\cot 20° \tan 60° + 1)$

$= 50 (2\dot{}747 \times \sqrt{3} + 1)$

$= 287\dot{}8... \text{ ft.} = 288 \text{ ft. nearly.}$

Ex. 4. *The elevation of a hill from a place P due East of it is* 45°, *and at a place Q due South of P, the elevation is* 30°. *If the distance PQ be* 400 *yds., find the height of the hill.*

A is the top of the hill, B is the point vertically below it on the ground. BP is due East, PQ is due South, so that BPQ is a right angle. Also ABP and ABQ are both right angles.

Now, $\dfrac{BQ}{AB} = \cot AQB = \cot 30° = \sqrt{3}$,

and $\dfrac{BP}{AB} = \cot APB = \cot 45° = 1$.

Hence, $BQ = AB\sqrt{3}$, $BP = AB$,
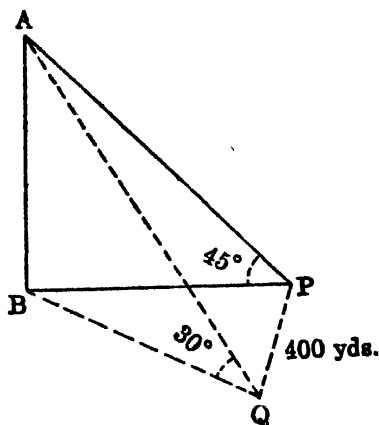
and $PQ^2 = BQ^2 - BP^2 = AB^2(3-1) = 2AB^2$.

$$\therefore \quad AB = \frac{PQ}{\sqrt{2}} = \frac{PQ}{2} \cdot \sqrt{2} = 200\sqrt{2} = 283 \text{ yds. nearly.}$$

Examples V

1. From the top of a tower by the seaside, 100 feet high, it was observed that the angle of depression of the bottom of a ship at anchor was 30°. Find the distance of the ship from the bottom of the tower.

2. Two straight roads, which cross one another, meet a river with straight course at angles 60° and 30° respectively. If it be 3 miles by the longer of the two roads, from the crossing to the river, how far is it by the shortest ? If there be a foot-path which goes the shortest way from the crossing to the river, what is the distance by it ?

3. Two poles are of equal height ; a person standing midway between the line joining their bases observes the

elevation of the poles to be 30°. After walking 40 feet towards one of them, he observes that the same pole now subtends an angle of 60°. Find their height and the distance between them.

4. A straight palm tree 60 feet high, is broken by the wind but not completely separated, and its upper part meets the ground at an angle of 30°. Find the distance of the point where the top of the tree meets the ground, from the root, and also the height at which the tree is broken.

5. Two posts are 120 ft. apart, and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds the angular elevations of their tops to be complementary₮ Find the height of the shorter post.   [*H. S. 1961 Com.*]

6. The Bally bridge subtends an angle of 45° at a given point at the edge of the river ; 800 yds. higher up, it subtends an angle of 30°. The course of the river here is straight and perpendicular to the bridge. Find the length of the bridge.

7. The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal straight line through the window ; find the height of the house, taking the breadth of the street to be 30 feet.

8. From an aeroplane vertically over a straight road, the angles of depression of two consecutive milestones are observed to be 45° and 60° ; find the height of the aeroplane.

9. From a ship sailing due South-East at the rate of 5 miles an hour, a light-house is observed to be 30° North of East, and after 4 hours, it is seen due North ; find the distance of the light-house from the final position of the ship.

10. The shadow of a tower standing on a level plane is found to be 40 feet longer when the sun's altitude is 45° than when it is 60°. Find the height of the tower.

11. From the lower window of a house the angular elevation of a church-steeple is found to be 45° and from a window 20 feet above, the elevation is 30°. How far is the church from the house ?

12. A light-house facing East sends out a fan-shaped beam of light extending from S. E. to N. E. An observer sailing due North, after meeting the light continues to see it for 10 $\sqrt{2}$ minutes. When leaving the fan of light, the ship is 10 miles from the light-house. Find the speed of the ship.

13. A pole 100 ft. high stands vertically at the centre of a horizontal equilateral triangle, each side of which subtends an angle of 60° at the top of the pole. Find the side of the triangle.

14. Two chimneys are of equal height. A person standing between them in the line joining their bases observes the elevation of the nearer one to be 60°. After walking 80 feet in a direction at right angles to the line joining their bases, he observes the elevations of the two to be 45° and 30° respectively. Find the height and the distance between them.

15. At the foot of a mountain the elevation of its summit is 45° ; after ascending 1 mile towards the mountain up an incline of 30°, the elevation changes to 60°. Find the height of the mountain.

16. From a station, two light-houses A and B are seen in directions North and 30° East of North respectively ; if A were one-third as far off as it really is, it would appear due West of B. If the distance of B from the station be 10 miles, find the distance of B from A.

17. A person walking along a straight road observes a tall tree standing in front of a tower, both being on the road before him. The elevation of the top of the tower is 45°, and that of the top of the tree 30° ; on advancing 100 feet he finds the tower and the tree to have the same elevation 60° ; supposing the height of the eye of the man to be 5 feet, find the height of the tower and of the tree.

18.   A man on the top of a rock rising on a seashore, observes a boat coming towards it at an angle of depression 30° ; 10 minutes later the angle of depression is 60°. The height of the rock being 4000 feet, find the speed of the boat in miles per hour.

19.   A person walking along a straight level road observes the elevation of the top of a hill to be 60° when he is nearest the hill, and after walking 200 yards in a direction perpendicular to the direction of the hill from this point, observes the elevation to be 30°. Find the approximate height of the hill.

20.   A square tower stands on a horizontal plane.   From a point in this plane, only three of its upper corners are visible, and their angles of elevation are 45°, 60°, 45°.   Find the ratio of the height of the tower to its breadth.

21.   Two wheels, the sum of whose radii is 10 feet, are placed flatly on a table with their centres at a distance of 20 ft.   An endless string, quite stretched, is partly wrapped round the wheels and crosses itself between them.   Show that the length of the string is nearly 76˙5 feet.

22.   On a still day, from a station A an airship is observed due North at an elevation of 60°, while from a station B it is observed due East at an elevation of 45°.   At this instant of observation, a parachute message is dropped from the airship, and the observer at A has to walk a mile to reach the message.   Find the distance between the two stations.

23.   From the foot of a column the angle of elevation of the top of a tower is 45° and from the top of the column the angle of depression of the bottom of the tower is 30°. A man walks 10 ft. from the bottom of the column towards the tower and notices the angle of elevation of its top to be 60°.   Find the height of the column.

COMPOUND ANGLES

33. *To Prove that*

sin (A + B) = sin A cos B + cos A sin B

cos (A + B) = cos A cos B − sin A sin B,

when *A* and *B* are *positive* and *acute* and $(A + B) < 90°$.

Let a revolving line starting from the position *OX* trace out an angle *XOY* = *A* and then revolving further, trace out an angle *YOZ* = *B* ; then ∠ *XOZ* = *A* + *B*.

In *OZ*, the *bounding line of the compound angle A + B*, take any point *P* and draw *PQ* and *PR* perpendiculars to *OX* and *OY* respectively ; also draw *RS* and *RT* perpendiculars to *OX* and *PQ* respectively.

From the right-angled △ *POQ*,

$$\sin (A + B) = \frac{PQ}{OP} = \frac{QT + TP}{OP} = \frac{RS + PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B + \cos TPR.\sin B.$$

Now, ∠ *TPR* = 90° − ∠ *TRP* = ∠ *TRO* = ∠ *ROS* = *A*.

$\therefore \quad \sin(A+B) = \sin A \cos B + \cos A \sin B.$

Again,

$$\cos(A+B) = \frac{OQ}{OP} = \frac{OS - QS}{OP} = \frac{OS - TR}{OP} = \frac{OS}{OP} - \frac{TR}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \cdot \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

34. *To prove that*

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B,$$

when A and B are *positive* and *acute*, and $A > B$.
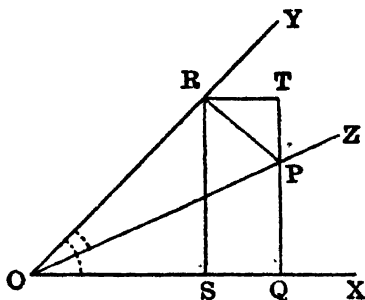
Let a revolving line start from the position OX and trace out an angle $XOY = A$ and then revolving back trace out an angle $YOZ = B$; then $\angle XOZ = A - B$.

In OZ, the *bounding line of the compound angle* $A - B$, take any point P, and draw PQ and PR perpendiculars to OX and OY respectively; and draw RS and RT perpendiculars to OX and QP produced respectively.

From the right-angled $\triangle POQ$,

$$\sin(A-B) = \frac{PQ}{OP} = \frac{TQ - PT}{OP} = \frac{RS - PT}{OP} = \frac{RS}{OP} - \frac{PT}{OP}$$

$$= \frac{RS}{OR} \cdot \frac{OR}{OP} - \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B - \cos TPR \cdot \sin B.$$

But $\angle TPR = 90° - \angle TRP = \angle YRT = \angle YOX = A.$

$\therefore$ $\sin (A - B) = \sin A \cos B - \cos A \sin B.$

Again,

$$\cos (A - B) = \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{RP} \cdot \frac{RP}{OP}$$

$$= \cos A \cos B + \sin TPR \cdot \sin B$$

$$= \cos A \cos B + \sin A \sin B.$$

Obs. In the above Geometrical proofs, it is assumed that the angles A, B, $A + B$ are all less than a right angle and that $A - B$ is positive. If the angles are not so restricted, the same method of proof (there being some modifications in the figures) will apply, due attention being paid to the signs of the quantities involved.[*]

Thus, the above formulæ are perfectly general.

Note 1. The sum or difference of two or more angles is called a *Compound angle*; such as $A + B$, $A - B$, $A + B + C$ etc.

The expansions $\sin (A \pm B)$ and $\cos (A \pm B)$ are generally called the "*Addition formulæ* or *Addition and Subtraction Theorems*".

Note 2. Assuming the truth of the above formulæ for acute angles, they can be shown to be true for angles of any magnitude, as follows :

Let us consider $\sin (A + B)$.

Let A and B be acute and $A + B < 90°$.

Let $A_1 = 90° + A$; $B_1 = B.$

Now, $\sin (A_1 + B_1) = \sin \{(90° + A) + B\} = \sin \{90° + (A + B)\}$

$\qquad = \cos (A + B) = \cos A \cos B - \sin A \sin B \qquad [\ by\ Art.\ 33\ ]$

$\qquad = \sin (90° + A) \cos B + \cos (90° + A) \sin B$

$\qquad = \sin A_1 \cos B_1 + \cos A_1 \sin B_1.$

---

[*]See *Appendix*, *Arts.* 2-4.   Also for alternative proofs of *Arts.* 33 and 34, see *Appendix*, *Art.* 5.

Again, let $A_2 = -A$, $B_2 = B$.

Then, $\sin (A_2 + B_2) = \sin (-A + B) = -\sin (A - B)$

$\qquad\qquad = -\sin A \cos B + \cos A \sin B$,   [*by Art. 34*]

$\qquad\qquad = \sin (-A) \cos B + \cos (-A) \sin B$

$\qquad\qquad = \sin A_2 \cos B_2 + \cos A_2 \sin B_2$.

Thus, the above formulæ remain true if any of the two angles is either increased by 90°, or has its sign changed.

In the same way it may be shown that the other three formulæ for $\cos (A + B)$, $\sin (A - B)$ and $\cos (A - B)$ will continue to hold good unchanged in form, if any of the two angles be either increased by 90° or has its sign changed.

Now starting from positive acute-angled values of A and B, combining the two processes of increasing one of the angles by 90°, and reversing the sign of any one, we can arrive at values of A and B of any magnitude, positive or negative and the four formulæ will still hold good.

Thus, the formulæ for $\sin (A \pm B)$ and $\cos (A \pm B)$ are perfectly general.

35. Ex. 1. *Find the values of*

$\qquad sin\ 75°,\ cos\ 75°,\ sin\ 15°\ and\ cos\ 15°.$

$\sin\ 75° = \sin (45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$\qquad = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}.$

$\cos\ 75° = \cos (45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$

$\qquad = \dfrac{1}{\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}} \cdot \dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}.$

$\sin\ 15° = \sin (45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$

and $\cos\ 15° = \cos (45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$;

therefore, substituting the values of $\sin 45°$, $\cos 45°$ etc. as before, we get

$$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{and} \quad \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Note. The values of $\sin 15°$ and $\cos 15°$ can also be deduced from the fact that

$$\sin 15° = \sin(90° - 75°) = \cos 75°$$
and $\quad \cos 15° = \cos(90° - 75°) = \sin 75°.$

Ex. 2. *Show that*

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
$$= \cos^2 B - \cos^2 A.$$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$
$$= \cos^2 B - \sin^2 A.$$

(i) Left side

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$
$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$
$= \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$
$= \sin^2 A - \sin^2 B$
$= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A.$

(ii) Left side

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$
$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
$= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$
$= \cos^2 A - \sin^2 B$
$= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A.$

Note. The results of Ex. 1 and Ex. 2 are very useful and should be carefully remembered.

36. *To prove that*

(i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}.$

(ii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}.$

We have

$$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Now, dividing the numerator and denominator by $\cos A \cos B$, we have,

$$\tan (A + B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Again,

$$\tan (A - B) = \frac{\sin (A - B)}{\cos (A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}.$$

Now, dividing the numerator and denominator by $\cos A \cos B$, we have, as before,

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}.$$

37. *To prove that*

(i) $\cot (A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}.$

(ii) $\cot (A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}.$

(i) $\cot (A + B) = \dfrac{\cos (A + B)}{\sin (A + B)} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}.$

Now, dividing the numerator and denominator by $\sin A \sin B$, we have,

$$\cot (A + B) = \frac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}}$$

$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

(ii) $\cot (A - B) = \dfrac{\cos (A - B)}{\sin (A - B)} = \dfrac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$.

Now, dividing the numerator and denominator by $\sin A \sin B$, we have, as before,

$$\cot (A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}.$$

38.   Ex. 1. *Find the values of tan 75° and tan 15°.*

$$\tan 75° = \tan (45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$\dfrac{1 + \dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1} = \dfrac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{3 - 1}$$

$$= \dfrac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

$$\tan 15° = \tan (45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$= \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} = \dfrac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$

$$= \dfrac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Ex. 2. *Show that*

(i) $\tan (45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$.

(ii) $\tan (45° - A) = \dfrac{1 - \tan A}{1 + \tan A}$.

(i) Left side $= \dfrac{\tan 45° + \tan A}{1 - \tan 45° \tan A} = \dfrac{1 + \tan A}{1 - \tan A}$.

(ii) This result follows similarly.

Ex. 3. *Show that*

$$\cot 2A + \tan A = \csc 2A. \qquad [\ C.\ U.\ 1947\ ]$$

Left side $= \dfrac{\cos 2A}{\sin 2A} + \dfrac{\sin A}{\cos A} = \dfrac{\cos 2A \cos A + \sin 2A \sin A}{\sin 2A \cos A}$

$= \dfrac{\cos (2A - A)}{\sin 2A \cos A} = \dfrac{\cos A}{\sin 2A \cos A} = \dfrac{1}{\sin 2A}$

$= \csc 2A.$

39. *To find the expansions of*

(i) sin (A + B + C)

(ii) cos (A + B + C)

(iii) tan (A + B + C).

(i) $\sin (A + B + C)$

$= \sin \{(A + B) + C\}$

$= \sin (A + B) \cos C + \cos (A + B) \sin C$

$= (\sin A \cos B + \cos A \sin B) \cos C$

$\qquad + (\cos A \cos B - \sin A \sin B) \sin C$

$= \sin A \cos B \cos C + \sin B \cos C \cos A$

$\qquad + \sin C \cos A \cos B - \sin A \sin B \sin C.$

Note 1. The expansion of sin $(A + B + C)$ can be easily put in the form

$\cos A \cos B \cos C\ (\tan A + \tan B + \tan C - \tan A \tan B \tan C).$

(ii) $\cos (A + B + C)$

$= \cos \{(A + B) + C\}$

$= \cos (A + B) \cos C - \sin (A + B) \sin C$

$= (\cos A \cos B - \sin A \sin B) \cos C$

$\qquad - (\sin A \cos B + \cos A \sin B) \sin C$

$= \cos A \cos B \cos C - \cos A \sin B \sin C$

$\qquad - \cos B \sin C \sin A - \cos C \sin A \sin B.$

Note 2. The expansion of cos $(A + B + C)$ can be easily put in the form

$\cos A \cos B \cos C\ (1 - \tan B \tan C - \tan C \tan A - \tan A \tan B).$

(iii) **tan (A + B + C)**

$$= \tan \{(A + B) + C\}$$

$$= \frac{\tan (A + B) + \tan C}{1 - \tan (A + B) \tan C}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \tan C}{1 - \dfrac{\tan A + \tan B}{1 - \tan A \tan B} \tan C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

Note 3 The expansion of $\tan (A + B + C)$ can also be obtained thus,

$$\tan (A + B + C) = \frac{\sin (A + B + C)}{\cos (A + B + C)}.$$

Now, write down the expansions of $\sin (A + B + C)$ and $\cos (A + B + C)$ and divide the numerator and denominator by $\cos A \cos B \cos C$ or simply write down the expansions of $\sin (A + B + C)$ and $\cos (A + B + C)$ as given in Notes 1 and 2.

Obs. Formulæ for the Trigonometrical functions of the sum of four, five or more angles can be similarly obtained.

Examples VI

Show that (*Ex. 1 to 20*) :—

1. (i) $\sin (A - B) = \frac{16}{65}$ and $\cos (A + B) = \frac{33}{65}$,

　　if A and B are acute and if $\sin A = \frac{3}{5}$, $\cos B = \frac{12}{13}$.

　(ii) $\cos 68° 20' \cos 8° 20' + \cos 81° 40' \cos 21° 40' = \frac{1}{2}$.

　(iii) $\sec (x - y) = \frac{85}{84}$, if $\sec x = \frac{17}{8}$, $\csc y = \frac{5}{4}$.

2. (i) $\sin A \sin (B - C) + \sin B \sin (C - A)$
　　　　　　　　　　$+ \sin C \sin (A - B) = 0$.

　(ii) $\cos A \sin (B - C) + \cos B \sin (C - A)$
　　　　　　　　　　$+ \cos C \sin (A - B) = 0$.

　(iii) $\sin (B + C) \sin (B - C) + \sin (C + A) \sin (C - A)$
　　　　　　　　　　$+ \sin (A + B) \sin (A - B) = 0$.

(iv) $\sin (\alpha - \theta) \sin (\beta - \gamma) + \sin (\beta - \theta) \sin (\gamma - \alpha)$
$$+ \sin (\gamma - \theta) \sin (\alpha - \beta) = 0.$$

3. $\cos (60° - A) \cos (30° - B) - \sin (60° - A) \sin (30° - B)$
$$= \sin (A + B).$$

4. (i) $\sin (n + 1) x \cos (n - 1)x - \cos (n + 1) x \sin (n - 1) x$
$$= \sin 2x.$$

(ii) $\sin 2\theta \cos \theta + \cos 2\theta \sin \theta$
$$= \sin 4\theta \cos \theta - \cos 4\theta \sin \theta.$$

5. $\dfrac{\sin B}{\sin A} = \dfrac{\sin (2A + B)}{\sin A} - 2 \cos (A + B).$

6. $\dfrac{\sin (B - C)}{\cos B \cos C} + \dfrac{\sin (C - A)}{\cos C \cos A} + \dfrac{\sin (A - B)}{\cos A \cos B} = 0.$

7. $\dfrac{\sin (B - C)}{\sin B \sin C} + \dfrac{\sin (C - A)}{\sin C \sin A} + \dfrac{\sin (A - B)}{\sin A \sin B} = 0.$

8. $\tan (A + B) \tan (A - B) = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}.$

9. $\tan^2 A - \tan^2 B = \dfrac{\sin (A + B) \sin (A - B)}{\cos^2 A \cos^2 B}.$

10. (i) $\dfrac{\tan (\alpha + \beta) - \tan \alpha}{1 + \tan (\alpha + \beta) \tan \alpha} = \tan \beta.$

(ii) If $A + B + C = \pi$ and $\cos A = \cos B \cos C$, show that $\tan A = \tan B + \tan C.$ $\hspace{1cm}$ [C. U. 1942]

11. $1 + \tan 2\theta \tan \theta = \sec 2\theta.$

12. $\cot \theta - \cot 2\theta = \operatorname{cosec} 2\theta.$

13. $\tan 20° + \tan 25° + \tan 25° \tan 20° = 1.$

14. (i) $\tan (45° + A) = \dfrac{\cos A + \sin A}{\cos A - \sin A}.$

(ii) $\sqrt{2} \sin (45° + A) = \sin A + \cos A.$

15. $\dfrac{\cos 8° + \sin 8°}{\cos 8° - \sin 8°} = \tan 53°.$

16. $\tan(45° + A)\tan(45° - A) = 1.$

17. $\tan(A + B) + \tan(A - B) = \dfrac{\sin 2A}{\cos^2 A - \sin^2 B}.$

18. $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}.$

19. $\cot(45° + x) = \dfrac{\cot x - 1}{\cot x + 1} = \dfrac{\cos x - \sin x}{\cos x + \sin x}.$

20. $\sec(x + y) = \dfrac{\sec x \sec y}{1 - \tan x \tan y}.$

21.   Find the expansions of
$\sin(A - B + C)$ and $\tan(A - B - C)$.

22.   Express $\cot(A + B + C)$ in terms of
$\cot A$, $\cot B$, $\cot C$.

23. (i) If $a\cos(x + a) = b\cos(x - a)$, prove that
$(a + b)\tan x = (a - b)\cot a$.

(ii) If $\sin a \sin \beta - \cos a \cos \beta + 1 = 0$, show that
$1 + \cot a \tan \beta = 0.$ [C. U. 1939]

(iii) If $A + B + C = \pi$ and $\cos A = \cos B \cos C$,
then $\cot B \cot C = \tfrac{1}{2}$.

24.   If $\tan \theta = \dfrac{a\sin x + b\sin y}{a\cos x + b\cos y}$,
then $a\sin(\theta - x) + b\sin(\theta - y) = 0$.

25.   An angle θ is divided into two parts a, β such that
$\tan a : \tan \beta = x : y$; prove that

$$\sin(a - \beta) = \frac{x - y}{x + y}\sin \theta.$$

26.   If $\cos(\beta - \gamma) + \cos(\gamma - a) + \cos(a - \beta) = -\tfrac{3}{2}$,
show that $\Sigma \cos a = 0$ and $\Sigma \sin a = 0$.

TRANSFORMATION OF PRODUCTS AND SUMS

40. Transformation of products into sums or differences.

We have from Arts. 33 and 34,

$$\sin A \cos B + \cos A \sin B = \sin (A + B) \quad \cdots \quad (1)$$
$$\sin A \cos B - \cos A \sin B = \sin (A - B). \quad \cdots \quad (2)$$

Adding (1) and (2), we get

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B). \quad \cdots \quad (3)$$

Subtracting (2) from (1), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B). \quad \cdots \quad (4)$$

Again, from Arts. 33 and 34, we have,

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad \cdots \quad (5)$$
$$\cos A \cos B + \sin A \sin B = \cos (A - B). \quad \cdots \quad (6)$$

Adding (5) and (6), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B). \quad \cdots \quad (7)$$

Subtracting (5) from (6), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \cdots \quad (8)$$

Thus, we have the following formula for transforming a *product* of two sines and cosines into the *sum* or the *difference* of two sines or two cosines.

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B). \quad \cdots \quad (I)$$
$$2 \cos A \sin B = \sin (A + B) - \sin (A - B). \quad \cdots \quad (II)$$
$$2 \cos A \cos B = \cos (A + B) + \cos (A - B). \quad \cdots \quad (III)$$
$$2 \sin A \sin B = \cos (A - B) - \cos (A + B). \quad \cdots \quad (IV)$$

41. Transformation of sums or differences into products.

Let $A + B = C$, and $A - B = D$,

then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$.

Making these substitutions for A and B in the results (3), (4), (7), (8) of Art. 40 and noting that the relation (8) can be written as

$$\cos(A+B) - \cos(A-B) = -2 \sin A \sin B$$
$$= 2 \sin A \sin(-B),$$

we have the following four formulæ for transforming the *sum* or the *difference* of **two sines** only or **two cosines** only into a product of sines and cosines.

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}. \quad \cdots \quad (I)$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}. \quad \cdots \quad (II)$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}. \quad \cdots \quad (III)$$

$$\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}. \quad \cdots \quad (IV)$$

Note. The following concise verbal statement of the above four formulæ is sometimes very convenient.

(i) $sine + sine = 2 \sin$ (½ sum) . $\cos$ (½ diff.),

(ii) $sine - sine = 2 \cos$ (½ sum) . $\sin$ (½ diff.).

(iii) $\cos + \cos = 2 \cos$ (½ sum) . $\cos$ (½ diff.).

(iv) $\cos - \cos = 2 \sin$ (½ sum) . $\sin$ (½ diff. reversed).

42. Ex. 1. *Prove that*

(i) $\cos 20° \cos 40° \cos 80° = \tfrac{1}{8}$.

(ii) $\cos 80° + \cos 40° - \cos 20° = 0$.

(i) Left side $= \frac{1}{2}.\cos 20° \, (2 \cos 40° \cos 80°)$

$\qquad\qquad = \frac{1}{2} \cos 20° \, (\cos 120° + \cos 40°)$

$\qquad\qquad = \frac{1}{2} \cos 20° \, (-\frac{1}{2} + \cos 40°)$

$\qquad\qquad = -\frac{1}{4} \cos 20° + \frac{1}{2} \cos 20° \cos 40°$

$\qquad\qquad = -\frac{1}{4} \cos 20° + \frac{1}{4} \, (\cos 60° + \cos 20°)$

$\qquad\qquad = -\frac{1}{4} \cos 20° + \frac{1}{4} \, (\frac{1}{2} + \cos 20°)$

$\qquad\qquad = \frac{1}{8}.$

(ii) Left side $= (\cos 80° + \cos 40°) - \cos 20°$

$\qquad\qquad = 2 \cos 60° \cos 20° - \cos 20°$

$\qquad\qquad = 2.\frac{1}{2} \cos 20° - \cos 20° = 0.$

Ex. 2.   *Show that*

$$\frac{\sin\theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos\theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta.$$

Numerator $= (\sin 5\theta + \sin\theta) + (\sin 4\theta + \sin 2\theta)$

$\qquad\qquad = 2 \sin 3\theta \cos 2\theta + 2 \sin 3\theta \cos\theta$

$\qquad\qquad = 2 \sin 3\theta \, (\cos 2\theta + \cos\theta) ;$

Denominator $= (\cos 5\theta + \cos\theta) + (\cos 4\theta + \cos 2\theta)$

$\qquad\qquad = 2 \cos 3\theta \cos 2\theta + 2 \cos 3\theta \cos\theta$

$\qquad\qquad = 2 \cos 3\theta \, (\cos 2\theta + \cos\theta).$

$\therefore$ left side $= \dfrac{2 \sin 3\theta \, (\cos 2\theta + \cos\theta)}{2 \cos 3\theta \, (\cos 2\theta + \cos\theta)} = \dfrac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta.$

Ex. 3.   *Express* 4 cos A cos B cos C *as the sum of four cosines.*

$4 \cos A \cos B \cos C = 2 \cos A.(2 \cos B \cos C)$

$\qquad\qquad = 2 \cos A \, \{\cos(B+C) + \cos(B-C)\}$

$\qquad\qquad = 2 \cos A \cos(B+C) + 2 \cos A \cos(B-C)$

$\qquad\qquad = \cos(A+B+C) + \cos(A-B-C)$

$\qquad\qquad\quad + \cos(A+B-C) + \cos(A-B+C).$

Ex. 4.   *Express as the product of three sines*

$\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C)$

$\qquad\qquad\qquad\qquad\qquad - \sin(A+B+C).$

Grouping together the first two terms and grouping together the last two terms, the given expression

$$= 2 \sin C \cos (B - A) + 2 \cos (A + B) \sin (- C)$$
$$= 2 \sin C \{\cos (B - A) - \cos (A + B)\}$$
$$= 2 \sin C (2 \sin B \sin A)$$
$$= 4 \sin A \sin B \sin C.$$

Examples VII

Prove that (*Ex. 1 to 17*) :—

1. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan \dfrac{A + B}{2} \cot \dfrac{A - B}{2}.$

2. $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot \dfrac{A + B}{2} \cot \dfrac{A - B}{2}.$

3. $\cos 20° + \cos 100° + \cos 140° = 0.$

4. $\sin \theta \sin (60° - \theta) \sin (60° + \theta) = \frac{1}{4} \sin 3\theta.$

5. $\cos \theta \cos (60° - \theta) \cos (60° + \theta) = \frac{1}{4} \cos 3\theta.$

6. $(\sin 3a + \sin a) \sin a + (\cos 3a - \cos a) \cos a = 0.$

7. $\cos (A - D) \sin (B - C) + \cos (B - D) \sin (C - A)$
    $$+ \cos (C - D) \sin (A - B) = 0.$$

8. $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}.$

9. $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}.$

10. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan \dfrac{A + B}{2}.$

11. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot \dfrac{A + B}{2}.$

12. $\dfrac{\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta.$

13. $\dfrac{\sin 2A + \sin 5A - \sin A}{\cos 2A + \cos 5A + \cos A} = \tan 2A.$

14. $\dfrac{\sin(a+\beta) - 2\sin a + \sin(a-\beta)}{\cos(a+\beta) - 2\cos a + \cos(a-\beta)} = \tan a.$

15. $\dfrac{\cos 7a + \cos 3a - \cos 5a - \cos a}{\sin 7a - \sin 3a - \sin 5a + \sin a} = \cot 2a.$

16. $\sin 2A + \sin 2B + \sin 2C - \sin 2(A+B+C)$
$$= 4\sin(B+C)\sin(C+A)\sin(A+B).$$

17. $\cos A + \cos B + \cos C + \cos(A+B+C)$
$$= 4\cos\frac{B+C}{2}\cos\frac{C+A}{2}\cos\frac{A+B}{2}.$$

18. If $\sin x = k\sin y$, prove that
$$\tan\tfrac{1}{2}(x-y) = \frac{k-1}{k+1}\tan\tfrac{1}{2}(x+y).$$

19. If $\cos x + \cos y = \tfrac{1}{3}$ and $\sin x + \sin y = \tfrac{1}{4}$, prove that
$$\tan\tfrac{1}{2}(x+y) = \tfrac{3}{4}.$$

20. If $x\cos a + y\sin a = k = x\cos\beta + y\sin\beta$, prove that
$$\frac{x}{\cos\tfrac{1}{2}(a+\beta)} = \frac{y}{\sin\tfrac{1}{2}(a+\beta)} = \frac{k}{\cos\tfrac{1}{2}(a-\beta)}.$$

21. If $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$, prove that
$$\tan\frac{\theta-\phi}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}.$$

22. Prove that $\dfrac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°} = \tan 35°.$

[*Note that $\sin\theta = \cos(90°-\theta)$ and $\cos\theta = \sin(90°\pm\theta)$.*]

23. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, then $\tan A . \tan B = \cot\tfrac{1}{2}(A+)$
$\times$ $\tan A + \tan B = \cot\tfrac{1}{2}(A+B).$ [*P. U. 1936*]

24. Prove that
$$\left(\frac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\frac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2\cot^n\frac{A-B}{2},$$
or zero, according as n is even or odd.   [*P. U. 1938*]

43. Trigonometrical ratios of angle 2A.

From Art. 33, we have,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B,$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

Putting $B = A$, in the first formula, we get

$$\sin 2A = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A. \qquad \cdots (1)$$

Putting $B = A$, in the second formula, we get

$$\cos 2A = \cos A.\cos A - \sin A.\sin A$$
$$= \cos^2 A - \sin^2 A \qquad \cdots (2)$$

$$= (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A \qquad \cdots (3)$$

and also $= \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1. \qquad \cdots (4)$

By Art. 36, $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}.$

Putting $B = A$, in the above formula, we get

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A.\tan A} = \frac{2 \tan A}{1 - \tan^2 A}. \qquad \cdots (5)$$

Similarly, putting $B = A$ in the value of $\cot (A + B)$ as given in Art. 37, we get $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}. \qquad \cdots (6)$

From formulæ (3) and (4), we obtain, by transposition,

$$1 + \cos 2A = 2 \cos^2 A \qquad \cdots \qquad \cdots (7)$$

$$1 - \cos 2A = 2 \sin^2 A. \qquad \cdots \qquad \cdots (8)$$

By division, $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A. \qquad \cdots \qquad \cdots (9)$

We may also note that

$$1 + \sin 2A = \cos^2 A + \sin^2 A + 2 \sin A \cos A$$
$$= (\cos A + \sin A)^2$$
$$1 - \sin 2A = \cos^2 A + \sin^2 A - 2 \sin A \cos A$$
$$= (\cos A - \sin A)^2.$$

Note. Since the addition formulæ are perfectly general (*i.e.*, true for all values of A and B), the above formulæ, being deduced from addition formulæ, are also perfectly general.

44.   Trigonometrical ratios of angle 3A.

$$\sin 3A = \sin (A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$
$$= \sin A (1 - 2 \sin^2 A) + \cos A.2 \sin A \cos A$$
$$[\textit{By Art. 43}]$$
$$= \sin A (1 - 2 \sin^2 A) + 2 \sin A (1 - \sin^2 A).$$

∴ **sin 3A = 3 sin A − 4 sin³A.**

$$\cos 3A = \cos (A + 2A) = \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A (2 \cos^2 A - 1) - \sin A.2 \sin A \cos A$$
$$= \cos A (2 \cos^2 A - 1) - 2 \cos A.\sin^2 A$$
$$= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A).$$

∴ **cos 3A = 4 cos³A − 3 cos A.**

$$\tan 3A = \tan (A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$
$$= \frac{\tan A + \dfrac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \cdot \dfrac{2 \tan A}{1 - \tan^2 A}}$$
$$= \frac{\tan A (1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A}.$$

$$\textbf{tan 3A} = \frac{\textbf{3 tan A} - \textbf{tan}^3\textbf{A}}{\textbf{1} - \textbf{3 tan}^2\textbf{A}}.$$

Obs. By a method similar to that of the previous article the trigonometrical ratios of any higher multiple of A can be expressed in terms of those of A.

45. Ex. 1. *Express* $\sin 2A$ *and* $\cos 2A$ *in terms of* $\tan A$.

$$\sin 2A = 2 \sin A \cos A = 2 \frac{\sin A}{\cos A} \cdot \cos^2 A$$

$$= 2 \tan A \cdot \frac{1}{\sec^2 A} = \frac{2 \tan A}{1 + \tan^2 A}.$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - \cos^2 A \cdot \frac{\sin^2 A}{\cos^2 A}$$

$$= \cos^2 A \left(1 - \frac{\sin^2 A}{\cos^2 A}\right) = \frac{1}{\sec^2 A} (1 - \tan^2 A)$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

Ex. 2. *Express* $\cos 4A$ *in terms of* $\cos A$.

Putting $\theta = 2A$, $\cos 4A = \cos 2\theta = 2 \cos^2 \theta - 1$

$$= 2 (\cos 2A)^2 - 1$$

$$= 2 (2 \cos^2 A - 1)^2 - 1$$

$$= 8 \cos^4 A - 8 \cos^2 A + 1.$$

Ex. 3. *Show that* $\dfrac{1 - \tan^2 (45° - A)}{1 + \tan^2 (45° - A)} = \sin 2A$.

Let $\theta = 45° - A$; then

$$\text{Left side} = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \dfrac{\sin^2 \theta}{\cos^2 \theta}}{1 + \dfrac{\sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$= \cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

$$= \cos (90° - 2A) = \sin 2A.$$

Examples VIII

Prove the following identities (*Ex. 1 to 24*) :—

1. $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A.$

2. $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A.$

3. $\cot A - \tan A = 2 \cot 2A.$

4. (i) $(2 \cos \theta + 1)(2 \cos \theta - 1) = 2 \cos 2\theta + 1.$

   (ii) $\tan \theta (1 + \sec 2\theta) = \tan 2\theta.$

5. $\dfrac{\cot A - \tan A}{\cot A + \tan A} = \cos 2A.$

6. $\tan A + \cot A = 2 \operatorname{cosec} 2A.$

7. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta.$

8. $\cos^6 \theta - \sin^6 \theta = \cos 2\theta \left(1 - \tfrac{1}{4} \sin^2 2\theta\right).$

9. $\cos^6 \theta + \sin^6 \theta = \tfrac{1}{4} \left(1 + 3 \cos^2 2\theta\right).$

10. $\dfrac{\sin^2 \alpha - \sin^2 \beta}{\sin \alpha \cos \alpha - \sin \beta \cos \beta} = \tan (\alpha + \beta).$

11. (i) $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan \theta.$ [*C. U. 1938*]

    (ii) $\dfrac{\sin \alpha - \sqrt{1 + \sin 2\alpha}}{\cos \alpha - \sqrt{1 + \sin 2\alpha}} = \cot \alpha.$ [α being positive and

acute, and the square root being taken with positive sign.]

12. $\dfrac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} - \dfrac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = 2 \tan 2\theta.$

13. (i) $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

    (ii) $\dfrac{\sin 4\theta}{\cos 2\theta} \cdot \dfrac{1 - \cos 2\theta}{1 - \cos 4\theta} = \tan \theta.$

14. (i) $\dfrac{\cos A - \sin A}{\cos A + \sin A} = \sec 2A - \tan 2A.$

    (ii) $\dfrac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} = 1 - \tfrac{1}{2} \sin 2\theta.$

15. $\cos^3 A \cos 3A + \sin^3 A \sin 3A = \cos^3 2A$.

16. (i) $4 (\cos^3 10° + \sin^3 20°) = 3 (\cos 10° + \sin 20°)$.

   (ii) $\cos 10° - \sqrt{3} \sin 10° = 2 \sin 20°$.

17. $\tan 3\theta - \tan 2\theta - \tan \theta = \tan 3\theta \tan 2\theta \tan \theta$.

18. $\dfrac{\cot A}{\cot A - \cot 3A} - \dfrac{\tan A}{\tan 3A - \tan A} = 1$.

19. $\dfrac{1}{\tan 3\theta - \tan \theta} - \dfrac{1}{\cot 3\theta - \cot \theta} = \cot 2\theta$.

20. $\sin 8\theta = 8 \sin \theta \cos \theta \cos 2\theta \cos 4\theta$.

21. (i) $\cos 5\theta = 16 \cos^5\theta - 20 \cos^3\theta + 5 \cos \theta$.

   (ii) $\sin 5\theta = 16 \sin^5\theta - 20 \sin^3\theta + 5 \sin \theta$.

22. (i) $\cot 3\theta = \dfrac{\cot^3\theta - 3 \cot \theta}{3 \cot^2\theta - 1}$.

   (ii) $\tan 4\theta = \dfrac{4 \tan \theta - 4 \tan^3\theta}{1 - 6 \tan^2\theta + \tan^4\theta}$.

23. (i) $\cos (120° - A) + \cos A + \cos (120° + A) = 0$.

   (ii) $\cos^2 (A - 120°) + \cos^2 A + \cos^2 (A + 120°) = \tfrac{3}{2}$.

24. $\dfrac{\tan 2^n\theta}{\tan \theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta) \cdots (1 + \sec 2^n\theta)$.

25.   If $\alpha = \tfrac{2}{7}\pi$, show that

   (i) $\cos \alpha \cos 2\alpha \cos 4\alpha = \tfrac{1}{8}$.

   (ii) $\cos \alpha + \cos 2\alpha + \cos 4\alpha = -\tfrac{1}{2}$.

   (iii) $\sin \alpha + \sin 2\alpha + \sin 4\alpha = \dfrac{\sqrt{7}}{8}$.

26.   If $\theta = \dfrac{\pi}{2^n + 1}$, prove that

$$2^n \cos \theta \cos 2\theta \cos 2^3\theta \cdots \cos 2^{n-1}\theta = 1.$$

27. (i) If $\tan x = b/a$, find the value of $a \cos 2x + b \sin 2x$.

(ii) If $\tan^2 x + 2 \tan x \tan 2y = \tan^2 y + 2 \tan y \tan 2x$, prove that each side $= 1$, or, else, $\tan x = \pm \tan y$.

28.   If $\tan^2 \theta = 1 + 2 \tan^2 \phi$, show that $\cos 2\phi = 1 + 2 \cos 2\theta$.

29. (i) If $2 \tan a = 3 \tan \beta$, prove that

$$\tan (a - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}. \qquad [\,C.\,U.\,1946\,]$$

(ii) If $\dfrac{\tan (a - \beta - \gamma)}{\tan (a - \beta + \gamma)} = \dfrac{\tan \gamma}{\tan \beta}$, show that

either, $\sin (\beta - \gamma) = 0$, *or*, $\sin 2a + \sin 2\beta + \sin 2\gamma = 0$.

30.   If a and β are acute angles and $\cos 2a = \dfrac{3 \cos 2\beta - 1}{3 - \cos 2\beta}$, show that $\tan a = \sqrt{2} \tan \beta$. $\qquad [\,C.\,U.\,1941\,]$

31.   If $\cos \theta = \frac{1}{2} (a + a^{-1})$, show that

(i) $\cos 2\theta = \frac{1}{2} (a^2 + a^{-2})$.

(ii) $\cos 3\theta = \frac{1}{2} (a^3 + a^{-3})$.

Show that (*Ex. 32 to 36*) :—

32. $\sin^4 \theta = \frac{3}{8} - \frac{1}{2} \cos 2\theta + \frac{1}{8} \cos 4\theta$.

33. $\cos^8 \theta + \sin^8 \theta = 1 - \sin^2 2\theta + \frac{1}{8} \sin^4 2\theta$.

34. $\tan \left(\dfrac{\pi}{4} + A \right) + \tan \left(\dfrac{\pi}{4} - A \right) = 2 \sec 2A$.

35. $\cos^3 \theta \dfrac{\sin 3\theta}{3} + \sin^3 \theta \dfrac{\cos 3\theta}{3} = \dfrac{\sin 4\theta}{4}$

36. $\cos 4x - \cos 4y$

$$= 8 (\cos x - \cos y)(\cos x + \cos y)(\cos x - \sin y) \\ \times (\cos x + \sin y).$$

SUBMULTIPLE ANGLES

46.   From the usual formulæ for multiple angles, namely

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$
$$1 + \cos 2A = 2 \cos^2 A \; ; \; 1 - \cos 2A = 2 \sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$
$$\sin 3A = 3 \sin A - 4 \sin^3 A$$
$$\cos 3A = 4 \cos^3 A - 3 \cos A$$
$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

putting $A = \tfrac{1}{2}\theta$ and $\tfrac{1}{3}\theta$ respectively we derive the following formulæ for submultiple angles :

$$\sin \theta = 2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$
$$\cos \theta = \cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta = 2 \cos^2 \tfrac{1}{2}\theta - 1 = 1 - 2 \sin^2 \tfrac{1}{2}\theta$$
$$1 + \cos \theta = 2 \cos^2 \tfrac{1}{2}\theta \; ; \; 1 - \cos \theta = 2 \sin^2 \tfrac{1}{2}\theta$$
$$\tan \theta = \frac{2 \tan \tfrac{1}{2}\theta}{1 - \tan^2 \tfrac{1}{2}\theta}$$
$$\sin \theta = 3 \sin \tfrac{1}{3}\theta - 4 \sin^3 \tfrac{1}{3}\theta$$
$$\cos \theta = 4 \cos^3 \tfrac{1}{3}\theta - 3 \cos \tfrac{1}{3}\theta$$
$$\tan \theta = \frac{3 \tan \tfrac{1}{3}\theta - \tan^3 \tfrac{1}{3}\theta}{1 - 3 \tan^2 \tfrac{1}{3}\theta}.$$

47.   Values of $\sin \tfrac{1}{2}\theta$ and $\cos \tfrac{1}{2}\theta$ in terms of $\cos \theta$.

From $\cos \theta = 2 \cos^2 \tfrac{1}{2}\theta - 1 = 1 - 2 \sin^2 \tfrac{1}{2}\theta$, we at   once deduce

$$\sin \tfrac{1}{2}\theta = \pm \sqrt{\tfrac{1}{2}(1 - \cos \theta)}$$
$$\cos \tfrac{1}{2}\theta = \pm \sqrt{\tfrac{1}{2}(1 + \cos \theta)}.$$

48. *Ambiguity of signs explained.*

When cos θ is given and not θ, θ and consequently $\frac{1}{2}\theta$ has a series of values as will be explained in Chapter XI. Thus, $\frac{1}{2}\theta$ may lie in any quadrant and sin $\frac{1}{2}\theta$ and cos $\frac{1}{2}\theta$ will then have corresponding signs.

If the quadrant in which $\frac{1}{2}\theta$ lies be known, for example, when θ is given along with cos θ, there is no ambiguity in choosing the proper signs of cos $\frac{1}{2}\theta$ and sin $\frac{1}{2}\theta$, as shown in the following example.

Ex. *Find sin* $22\frac{1}{2}°$ *and cos* $22\frac{1}{2}°$.

$$\sin 22\tfrac{1}{2}° = + \sqrt{\tfrac{1}{2}(1 - \cos 45°)} = \sqrt{\frac{1}{2}\left(1 - \frac{1}{\sqrt 2}\right)} = \frac{1}{2}\sqrt{2 - \sqrt 2}$$

$$\cos 22\tfrac{1}{2}° = + \sqrt{\tfrac{1}{2}(1 + \cos 45°)} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt 2}\right)} = \frac{1}{2}\sqrt{2 + \sqrt 2}.$$

49. Values of sin $\frac{1}{2}\theta$ and cos $\frac{1}{2}\theta$ in terms of sin θ.

We know that sin $\theta = 2$ sin $\frac{1}{2}\theta$ cos $\frac{1}{2}\theta$

and $\qquad 1 = \cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta.$

Therefore, $1 + \sin \theta = (\cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta)^2,$

and $\quad 1 - \sin \theta = (\cos \frac{1}{2}\theta - \sin \frac{1}{2}\theta)^2.$

Hence, cos $\frac{1}{2}\theta + $ sin $\frac{1}{2}\theta = \pm \sqrt{1 + \sin \theta}$

$\qquad$ cos $\frac{1}{2}\theta - $ sin $\frac{1}{2}\theta = \pm \sqrt{1 - \sin \theta}.$

Thus, **cos $\frac{1}{2}\theta = \pm \frac{1}{2}\sqrt{1 + \sin \theta} \pm \frac{1}{2}\sqrt{1 - \sin \theta}$**

and sin $\frac{1}{2}\theta = \pm \frac{1}{2}\sqrt{1 + \sin \theta} \mp \frac{1}{2}\sqrt{1 - \sin \theta}.$

50. *Ambiguity of signs explained.*

As before, when sin θ is given, and not θ, θ has a series of values for the given value of sin θ as will be explained in Chapter XI; $\frac{1}{2}\theta$ may therefore lie in any one of two possible quadrants.

$$\cos \tfrac{1}{2}\theta + \sin \tfrac{1}{2}\theta = \sqrt{2} \sin (\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)$$

and $\cos \tfrac{1}{2}\theta - \sin \tfrac{1}{2}\theta = \sqrt{2} \sin (\tfrac{1}{4}\pi - \tfrac{1}{2}\theta)$

will have their signs determined accordingly.

Thus, $\sin \tfrac{1}{2}\theta$ and $\cos \tfrac{1}{2}\theta$ will be definitely known.

Ex. *Find sin* 15° *and cos* 15°.

We have, $\cos 15° + \sin 15° = + \sqrt{1 + \sin 30°} = \sqrt{1 + \tfrac{1}{2}}$

$$\cos 15° - \sin 15° = + \sqrt{1 - \sin 30°} = \sqrt{1 - \tfrac{1}{2}}.$$

[$\cos 15° - \sin 15° = \sqrt{2} \sin (\tfrac{1}{4}\pi - 15°)$ and is clearly positive.]

Thus, $\cos 15° = \tfrac{1}{2} (\sqrt{\tfrac{3}{2}} + \sqrt{\tfrac{1}{2}}) = \dfrac{\sqrt{3} + 1}{2 \sqrt{2}}$

$$\sin 15° = \tfrac{1}{2} (\sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}}) = \dfrac{\sqrt{3} - 1}{2 \sqrt{2}}.$$

51. tan ½θ in terms of tan θ.

From the formula, $\tan \theta = \dfrac{2 \tan \tfrac{1}{2}\theta}{1 - \tan^2 \tfrac{1}{2}\theta}$,

i e., $\tan \theta \tan^2 \tfrac{1}{2}\theta + 2 \tan \tfrac{1}{2}\theta - \tan \theta = 0$,

we easily deduce

$$\tan \tfrac{1}{2}\theta = \dfrac{-1 + \sqrt{1 + \tan^2 \theta}}{\tan \theta}.$$

The reason of the ambiguity is similar to those of the previous cases.

52. Ratios of ⅓θ from those of θ.

By solving the cubic equation

$$\sin \theta = 3 \sin \tfrac{1}{3}\theta - 4 \sin^3 \tfrac{1}{3}\theta \qquad \cdots \qquad \cdots \quad (1)$$

we get $\sin \tfrac{1}{3}\theta$, if $\sin \theta$ be known.

Similarly, by solving the cubic equations

$$\cos \theta = 4 \cos^3 \tfrac{1}{3}\theta - 3 \cos \tfrac{1}{3}\theta \qquad \cdots \qquad \cdots \quad (2)$$

and $\quad \tan \theta = \dfrac{3 \tan \frac{1}{3}\theta - \tan^3 \frac{1}{3}\theta}{1 - 3 \tan^2 \frac{1}{3}\theta} \qquad \cdots \quad (3)$

we derive values of $\cos \frac{1}{3}\theta$ from those of $\cos \theta$, and of $\tan \frac{1}{3}\theta$ from those of $\tan \theta$ respectively.

53. Ratios of 18° and 36°.

Let $\theta = 18°$; then $5\theta = 90°$; $\therefore 2\theta = 90° - 3\theta$.

$\therefore \quad \sin 2\theta = \cos 3\theta$, or, $2 \sin \theta \cos \theta = \cos \theta (4 \cos^2\theta - 3)$.

As $\cos \theta$ (*i.e.* $\cos 18°$) is not zero, we have

$\qquad 2 \sin \theta = 4 \cos^2\theta - 3 = 1 - 4 \sin^2\theta,$

or, $\quad 4 \sin^2\theta + 2 \sin \theta - 1 = 0.$

$\therefore \ \sin \theta = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{(\pm \sqrt{5}-1)}{4}.$

Now, as θ here is a positive acute angle, therefore, rejecting the negative value, we get

$\qquad \sin 18° = \frac{1}{4}(\sqrt{5}-1).$
$\qquad \cos 18° = + \sqrt{1 - \sin^2 18°} = \frac{1}{4}(\sqrt{10+2\sqrt{5}}).$
$\qquad \cos 36° = 1 - 2 \sin^2 18° = \frac{1}{4}(\sqrt{5}+1).$
$\qquad \sin 36° = \sqrt{1 - \cos^2 36°} = \frac{1}{4}(\sqrt{10-2\sqrt{5}}).$

Note. Since 54° and 36° are complementary and 72° and 18° are complementary, from the above values we easily get the trigonometrical ratios of 54° and 72°.

54. Ratios of 3° and multiples of 3°.

$\sin 3° = \sin (18° - 15°) = \sin 18° \cos 15° - \cos 18° \sin 15°$
$\qquad = \frac{1}{16}(\sqrt{5}-1)(\sqrt{6}+\sqrt{2}) - \frac{1}{8}(\sqrt{3}-1)(\sqrt{5+\sqrt{5}}),$

on substituting the values of $\sin 18°$, $\cos 15°$, etc.

Similarly,

$\qquad \cos 3° = \frac{1}{8}(\sqrt{3}+1)(\sqrt{5+\sqrt{5}}) + \frac{1}{16}(\sqrt{6}-\sqrt{2})(\sqrt{5}-1).$

From a knowledge of the ratios of 3°, 15°, 18°, 30°, 36° and 45°, we can deduce the ratios for all angles which are multiples of 3°, (for, 6° = 36° - 30°; 9° = 45° - 36°;

$12° = 30° - 18°$; $21° = 36° - 15°$; etc.). For angles greater than $45°$, the ratios may be deduced from those of their complements which are less than $45°$.

Ex. *Show that*

$$sin \; x = 2^n \; cos \; \frac{x}{2} \; cos \; \frac{x}{2^2} \; cos \; \frac{x}{2^3} \; \cdots\cdots\cdots \; cos \; \frac{x}{2^n} \; sin \; \frac{x}{2^n}.$$

We have, $sin \; x = 2 \; sin \dfrac{x}{2} \; cos \dfrac{x}{2}$

$$sin \; \frac{x}{2} = 2 \; sin \frac{x}{2^2} \; cos \frac{x}{2^2}$$

$$sin \; \frac{x}{2^2} = 2 \; sin \frac{x}{2^3} \; cos \frac{x}{2^3}$$

$$\cdots \qquad \cdots \qquad \cdots$$

Similarly, $sin \dfrac{x}{2^{n-1}} = 2 \; sin \dfrac{x}{2^n} \; cos \dfrac{x}{2^n}.$

Hence, $sin \; x = 2^n \; cos \dfrac{x}{2} \; cos \dfrac{x}{2^2} \; cos \dfrac{x}{2^3} \cdots\cdots \; cos \dfrac{x}{2^n} \; sin \dfrac{x}{2^n}.$

Examples IX

Prove that (*Ex. 1 to 14*):—

1. $\dfrac{1 - \cos A}{\sin A} = \tan \dfrac{A}{2}.$ **2.** $\dfrac{1 + \cos A}{\sin A} = \cot \dfrac{A}{2}.$

3. $\left(\sin \dfrac{A}{2} \pm \cos \dfrac{A}{2} \right)^2 = 1 \pm \sin A.$

4. $\sec \theta + \tan \theta = \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\theta).$ [*C. U. 1939*]

5. (i) $\dfrac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \dfrac{\theta}{2}.$

    (ii) $\dfrac{\sin \tfrac{1}{2}a - \sqrt{1 + \sin a}}{\cos \tfrac{1}{2}a - \sqrt{1 + \sin a}} = \cot \dfrac{a}{2}$ where $0 < a < \pi,$

    and the square root is taken with positive sign.

6. (i) $\dfrac{1 + \sin x}{1 - \sin x} = \tan^2 \left(\dfrac{\pi}{2} + \dfrac{x}{2} \right).$

    (ii) $\dfrac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta} = \tan^2 \tfrac{1}{2}\theta.$

7. (i) $\dfrac{1 + \tan \frac{1}{2}A}{1 - \tan \frac{1}{2}A} = \dfrac{1 + \sin A}{\cos A}$.

(ii) $\cot \beta = \frac{1}{2} (\cot \frac{1}{2}\beta - \tan \frac{1}{2}\beta)$.

8. (i) $\dfrac{\sin 2\theta}{1 + \cos 2\theta} \cdot \dfrac{\cos \theta}{1 + \cos \theta} = \tan \dfrac{\theta}{2}$.

(ii) $8 \sin^4 \frac{1}{2}\theta - 8 \sin^2 \frac{1}{2}\theta + 1 = \cos 2\theta$.

9. $\sin \theta = \dfrac{2 \tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}$.	10. $\cos \theta = \dfrac{1 - \tan^2 \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta}$.

11. $(\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 4 \cos^2 \frac{1}{2} (x - y)$.

12. $\tan 6° \tan 42° \tan 66° \tan 78° = 1$.

13. $\tan 7\frac{1}{2}° = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

14. $2 \cos \frac{1}{16}\pi = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$.

15. (i) If $\tan \dfrac{\theta}{2} = \sqrt{\dfrac{1 - e}{1 + e}} \cdot \tan \dfrac{\phi}{2}$, show that

$$\cos \phi = \dfrac{\cos \theta - e}{1 - e \cos \theta}.$$

(ii) If $\tan \theta = \dfrac{\sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$, show that one of the values of $\tan \frac{1}{2}\theta$ is $\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta$.

16. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, find the value of $\cos (\alpha + \beta)$.

17. (i) Prove that $2 \sin \frac{1}{2}A = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}$, and determine which are the correct signs when $270° > A > 180°$.	[B. H. U. I., 1931]

(ii) If $\theta = 240°$, is the following statement correct ?

$2 \sin \frac{1}{2}\theta = \sqrt{1 + \sin \theta} - \sqrt{1 - \sin \theta}$.

If not, how must it be modified ?

18. If $A = 320°$, prove that

$$\tan \dfrac{A}{2} = \dfrac{-1 + \sqrt{1 + \tan^2 A}}{\tan A}.$$

TRIGONOMETRICAL IDENTITIES

55. Many interesting identities involving functions of three or more angles can be established when there exists a relation among the angles. The most important of these identities are those in which the three angles are connected by the relation that their sum is equal to two right angles. In establishing this latter kind of identities, it will be necessary to make frequent use of the properties of supplementary and complementary angles.

Thus, since $A + B + C = \pi$,

$$\therefore \qquad B + C = \pi - A.$$

$$\therefore \qquad \sin (B + C) = \sin (\pi - A) = \sin A.$$

Similarly, $\sin (C + A) = \sin B$; $\sin (A + B) = \sin C$.

Again, $\cos (B + C) = \cos (\pi - A) = -\cos A.$

Similarly, $\cos (C + A) = -\cos B$; $\cos (A + B) = -\cos C.$

$$\tan (B + C) = \tan (\pi - A) = -\tan A.$$

Similarly, $\tan (C + A) = -\tan B$; $\tan (A + B) = -\tan C.$

Again, since, $\dfrac{A}{2} + \dfrac{B}{2} + \dfrac{C}{2} = \dfrac{\pi}{2}$,

$$\therefore \quad \sin \left(\dfrac{B}{2} + \dfrac{C}{2} \right) = \sin \left(\dfrac{\pi}{2} - \dfrac{A}{2} \right) = \cos$$

Similarly, $\sin \left(\dfrac{C}{2} + \dfrac{A}{2} \right) = \cos \dfrac{B}{2}$;

$$\sin \left(\dfrac{A}{2} + \dfrac{B}{2} \right) = \cos \dfrac{C}{2}.$$

Again, $\cos \left(\dfrac{B}{2} + \dfrac{C}{2} \right) = \cos \left(\dfrac{\pi}{2} - \dfrac{A}{2} \right) = \sin$

Similarly, $\cos\left(\dfrac{C}{2} + \dfrac{A}{2}\right) = \sin\dfrac{B}{2}$;

$\qquad\cos\left(\dfrac{A}{2} + \dfrac{B}{2}\right) = \sin\dfrac{C}{2}$.

$\qquad\tan\left(\dfrac{B}{2} + \dfrac{C}{2}\right) = \tan\left(\dfrac{\pi}{2} - \dfrac{A}{2}\right) = \cot\dfrac{A}{2}$

Similarly, $\tan\left(\dfrac{C}{2} + \dfrac{A}{2}\right) = \cot\dfrac{B}{2}$;

$\qquad\tan\left(\dfrac{A}{2} + \dfrac{B}{2}\right) = \cot\dfrac{C}{2}$.

56.   Ex. 1.   *If* A+B+C=π, *prove that*

$\qquad$ sin 2A + sin 2B + sin 2C = 4 sin A sin B sin C.

$\qquad\qquad$ [*C. U. 1931, '33, '35 ; H. S. '61, Comp.*]

Left side $= (\sin 2A + \sin 2B) + \sin 2C$

$\qquad = 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$\qquad = 2\sin C\cos(A-B) + 2\sin C\cos C$

$\qquad\qquad\qquad\qquad$ [∵ $A+B+C=\pi$.]

$\qquad = 2\sin C\,[\cos(A-B) + \cos C]$

$\qquad = 2\sin C\,[\cos(A-B) - \cos(A+B)]$

$\qquad\qquad\qquad\qquad$ [∵ $A+B+C=\pi$.]

$\qquad = 2\sin C\,.\,2\sin A\sin B$

$\qquad = 4\sin A\sin B\sin C$.

Ex. 2.   *If* A+B+C=π, *prove that*

$\quad$ cos 2A + cos 2B + cos 2C = − 4 cos A cos B cos C − 1.

Left side $= (\cos 2A + \cos 2B) + \cos 2C$

$\qquad = 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1$

$\qquad = -2\cos C\cos(A-B) + 2\cos^2 C - 1$

$\qquad\qquad\qquad\qquad$ [∵ $A+B+C=\pi$.]

$\qquad = -2\cos C\,[\cos(A-B) - \cos C] - 1$

$\qquad = -2\cos C\,[\cos(A-B) + \cos(A+B)] - 1$

$\qquad\qquad\qquad\qquad$ [∵ $A+B+C=\pi$.]

$\qquad = -2\cos C\,.\,2\cos A\cos B - 1$

$\qquad = -4\cos A\cos B\cos C - 1$.

Ex. 3. *If* A+B+C = π, *prove that*

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

[*C. U. 1910, '29* ; *H. S. '60, Comp.*]

Left side $=(\sin A+\sin B)+\sin C$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$\left[\because\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right]$$

$$\left[\because\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right]$$

$$= 2\cos\frac{C}{2}\cdot2\cos\frac{A}{2}\cos\frac{B}{2}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Ex. 4. *If* A+B+C = π, *prove that*

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Left side $=(\cos A+\cos B)+\cos C$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1$$

$$\left[\because\ \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\right]$$

$$= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \sin \frac{C}{2} \right] + 1$$

$$= 2 \sin \frac{C}{2} \left[\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right] + 1$$

$$\left[\because \frac{A}{2} + \frac{B}{2} + \frac{C}{2} \right.$$

$$= 2 \sin \frac{C}{2} \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2} + 1$$

$$= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

Ex. 5. *If* A+B+C=π, *prove that*

   tan A+tan B+tan C = tan A tan B tan C.

[*H. S. 1961*]

Since, $B + C = \pi - A,$

∴ $\tan (B+C) = \tan (\pi - A).$

∴ $\dfrac{\tan B + \tan C}{1 - \tan B \tan C} = - \tan A,$

i.e., $\tan B + \tan C = - \tan A (1 - \tan B \tan C)$

$$= - \tan A + \tan A \tan B \tan C.$$

∴ $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

Otherwise :

   $\tan (A + B + C) = \tan \pi = 0.$

∴ $\dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B} = 0.$

Since, the fraction is zero, numerator must be zero.

∴ $\tan A + \tan B + \tan C - \tan A \tan B \tan C = 0,$

i.e., $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

Ex. 6. *If* A+B+C=π, *prove that*

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

[*C. U. 1936, '39*]

Since, $A+B+C=\pi$, $\therefore$ $\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}$.

$\therefore$ $\tan\left(\dfrac{B}{2}+\dfrac{C}{2}\right)=\tan\left(\dfrac{\pi}{2}-\dfrac{A}{2}\right)$.

$\therefore$ $\dfrac{\tan\dfrac{B}{2}+\tan\dfrac{C}{2}}{1-\tan\dfrac{B}{2}\tan\dfrac{C}{2}}=\cot\dfrac{A}{2}=\dfrac{1}{\tan\dfrac{A}{2}}$,

or, $\tan\dfrac{A}{2}\left(\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right)=1-\tan\dfrac{B}{2}\tan\dfrac{C}{2}$.

On simplification, the required result follows.

Otherwise :

$$1\Big/\tan\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\cot\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\cot\dfrac{\pi}{2}$$

$$\dfrac{1-\tan\dfrac{B}{2}\tan\dfrac{C}{2}-\tan\dfrac{C}{2}\tan\dfrac{A}{2}-\tan\dfrac{A}{2}\tan\dfrac{B}{2}}{\tan\dfrac{A}{2}+\tan\dfrac{B}{2}+\tan\dfrac{C}{2}-\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}}=0.$$

Now the value of the fraction being zero, its numerator must be zero.

$\therefore$ $1-\tan\dfrac{B}{2}\tan\dfrac{C}{2}-\tan\dfrac{C}{2}\tan\dfrac{A}{2}-\tan\dfrac{A}{2}\tan\dfrac{B}{2}=0$,

whence the required result follows.

Ex. 7.   *If* $A+B+C=\pi$, *prove that*

$$\cos\dfrac{A}{2}+\cos\dfrac{B}{2}+\cos\dfrac{C}{2}=4\cos\dfrac{\pi-A}{4}\cos\dfrac{\pi-B}{4}\cos\dfrac{\pi-C}{4}.$$

Right side $=2\cos\dfrac{\pi-A}{4}\left[2\cos\dfrac{\pi-B}{4}\cos\dfrac{\pi-C}{4}\right]$

$=2\cos\dfrac{\pi-A}{4}\left[\cos\dfrac{2\pi-(B+C)}{4}+\cos\dfrac{B-C}{4}\right]$

$=2\cos\dfrac{\pi-A}{4}\left[\cos\dfrac{\pi+A}{4}+\cos\dfrac{B-C}{4}\right]$

$[\because 2\pi-(B+C)=\pi+\pi-(B+C)=\pi+A$, since. $A+B+C=\pi$. $]$

$$= 2 \cos \frac{\pi - A}{4} \ \cos \frac{\pi + A}{4} + 2 \cos \frac{\pi - A}{4} \ \cos \frac{B - C}{4}$$

$$= \left(\cos \frac{\pi}{2} + \cos \frac{A}{2} \right) + 2 \cos \frac{B + C}{4} \cos \frac{B - C}{4}$$

$$[\because A + B + C = \pi.]$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}.$$

Note.   Since, $\cos \tfrac{1}{4} (\pi - A) = \sin \{ \tfrac{1}{2}\pi - \tfrac{1}{4} (\pi - A) \} = \sin \tfrac{1}{4} (\pi + A)$
           and $\cos \tfrac{1}{4} (\pi - A) = \cos \tfrac{1}{4} (A + B + C - A) = \cos \tfrac{1}{4} (B + C).$

$\therefore$ we have also, $\cos \tfrac{1}{2}A + \cos \tfrac{1}{2}B + \cos \tfrac{1}{2}C$

$$= 4 \sin \tfrac{1}{4} (\pi + A) \sin \tfrac{1}{4} (\pi + B) \sin \tfrac{1}{4} (\pi + C)$$

$$= 4 \cos \tfrac{1}{4} (B + C) \cos \tfrac{1}{4} (C + A) \cos \tfrac{1}{4} (A + B).$$

Ex. 8.   *If* $A + B + C = \pi$, *prove that*

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

$$[\ C. \ U. \ 1932, \ '37, \ '47 \ ]$$

$\cos^2 A + \cos^2 B + \cos^2 C$

$$= \tfrac{1}{2} (2 \cos^2 A + 2 \cos^2 B) + \cos^2 C$$

$$= \tfrac{1}{2} (1 + \cos 2A + 1 + \cos 2B) + \cos^2 C$$

$$= 1 + \tfrac{1}{2} (\cos 2A + \cos 2B) + \cos^2 C$$

$$= 1 + \cos (A + B) \cos (A - B) + \cos C. \ \cos C$$

$$= 1 - \cos C \cos (A - B) - \cos C \cos (A + B)$$

$$[\because A + B = \pi - C.]$$

$$= 1 - \cos C \ [\ \cos (A - B) + \cos (A + B) \ ]$$

$$= 1 - \cos C \ [\ 2 \cos A \cos B \ ]$$

$$= 1 - 2 \cos A \cos B \cos C,$$

whence the required result follows.

Ex. 9.   *Show that*

$$tan \ (\beta - \gamma) + tan \ (\gamma - a) + tan \ (a - \beta)$$
$$= tan \ (\beta - \gamma) \ tan \ (\gamma - a) \ tan \ (a - \beta).$$

Let $A = \beta - \gamma, B = \gamma - a, C = a - \beta$;

    then $A + B + C = \beta - \gamma + \gamma - a + a - \beta = 0.$

$\therefore$ $\tan (A + B + C) = \tan 0 = 0.$

$\therefore$ $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$

Now, substituting the values for A, B, C, the required result follows.

Ex. 10.   *If $x + y + z = xyz$, prove that*
$$x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1 - y^2) = 4xyz.$$

Putting $x = \tan a, y = \tan \beta, z = \tan \gamma$, in the given relation, we have

        $\tan a + \tan \beta + \tan \gamma = \tan a \tan \beta \tan \gamma.$

$\therefore$ by transposition,

        $\tan a (1 - \tan \beta \tan \gamma) = -(\tan \beta + \tan \gamma).$

i.e., $\tan a = - \dfrac{\tan \beta + \tan \gamma}{1 - \tan \beta \tan \gamma} = - \tan (\beta + \gamma).$

$\therefore$ $a = \pi - (\beta + \gamma).$ $\therefore$ $a + \beta + \gamma = \pi.$ $\therefore$ $2a + 2\beta + 2\gamma = 2\pi.$

$\therefore$ $\tan (2a + 2\beta + 2\gamma) = \tan 2\pi = 0.$

Therefore, as in Ex. 5 above,

        $\tan 2a + \tan 2\beta + \tan 2\gamma = \tan 2a \tan 2\beta \tan 2\gamma.$

Now, expressing $\tan 2a, \tan 2\beta, \tan 2\gamma$ in terms of $\tan a$, $\tan \beta, \tan \gamma$ and substituting x, y, z, for them, we get,

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{8xyz}{(1 - x^2)(1 - y^2)(1 - z^2)}.$$

On simplification, the required result follows.

Examples X

If $A + B + C = \pi$, prove that (*Ex. 1 to 16*) :—

$$\frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$
$$[H. S. 1962]$$

2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$

3. $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \cot \dfrac{A}{2} \cot \dfrac{B}{2} \cot \dfrac{C}{2}$.

4. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$.

5. $(\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B)$
$$= \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

6. $\dfrac{\cot B + \cot C}{\tan B + \tan C} + \dfrac{\cot C + \cot A}{\tan C + \tan A} + \dfrac{\cot A + \cot B}{\tan A + \tan B} = 1.$

7. $\sin \dfrac{A}{2} + \sin \dfrac{B}{2} + \sin \dfrac{C}{2}$

$$= 1 + 4 \sin \dfrac{\pi - A}{4} \sin \dfrac{\pi - B}{4} \sin \dfrac{\pi - C}{4}$$

$$= 1 + 4 \sin \dfrac{B + C}{4} \sin \dfrac{C + A}{4} \sin \dfrac{A + B}{4}.$$

8. $\cos^2 2A + \cos^2 2B + \cos^2 2C$
$$= 1 + 2 \cos 2A \cos 2B \cos 2C.$$

9. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.$

10. $\sin^2 \dfrac{A}{2} + \sin^2 \dfrac{B}{2} + \sin^2 \dfrac{C}{2} = 1 - 2 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}.$

11. $\dfrac{\cos A}{\sin B \sin C} + \dfrac{\cos B}{\sin C \sin A} + \dfrac{\cos C}{\sin A \sin B} = 2.$

$$[\; C.\ U.\ 1949\;]$$

12. $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}.$

13. $\sin (B + C - A) + \sin (C + A - B) + \sin (A + B - C)$
$$= 4 \sin A \sin B \sin C.$$

14. $\sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B)$
$$= 4 \sin \dfrac{B - C}{2} \sin \dfrac{C - A}{2} \sin \dfrac{A - B}{2}.$$

15. $\cos^2 A + \cos^2 B + 2 \cos A \cos B \cos C = \sin^2 C.$

16. $\cos \dfrac{A}{2} \cos \dfrac{B - C}{2} + \cos \dfrac{B}{2} \cos \dfrac{C - A}{2} + \cos \dfrac{C}{2} \cos \dfrac{A - B}{2}$
$$= \sin A + \sin B + \sin C.$$

17. If $\alpha + \beta + \gamma = \frac{1}{2}\pi$, prove that

(i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta\sin\gamma = 1$.

[*C. U. 1943*]

(ii) $\tan\beta\tan\gamma + \tan\gamma\tan\alpha + \tan\alpha\tan\beta = 1$.

18. If A, B, C, D are the angles of a quadrilateral, prove that

(i) $\dfrac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$

$\qquad = \tan A \tan B \tan C \tan D$.

(ii) $\cos A + \cos B + \cos C + \cos D$

$\qquad = 4\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(C+A)$.

19. Show that

(i) $\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta)$

$\qquad = 1 + 2\cos(\beta - \gamma)\cos(\gamma - \alpha)\cos(\alpha - \beta)$.

(ii) $\sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta\cos(\alpha + \beta) = \sin^2(\alpha + \beta)$.

(iii) $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\theta\cos(\alpha + \theta)$

is independent of θ.

20. (i) If $\alpha + \beta = \gamma$, show that

$\qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma$.

[*C. U. 1940*]

(ii) If $\alpha + \beta + \gamma = 2\pi$, show that

$\qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 1$.

21. If $\cos(A+B)\sin(C+D) = \cos(A-B)\sin(C-D)$, show that

$\qquad \cot A \cot B \cot C = \cot D$.

22. If $A + B + C = 2S$, prove that

(i) $\sin(S-A) + \sin(S-B) + \sin(S-C) - \sin S$

$\qquad\qquad\qquad = 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

(ii) $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1$

$\qquad = 4\cos S\cos(S-A)\cos(S-B)\cos(S-C)$.

23. If $A + B + C = n\pi$ (n being zero or an integer),
then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

24. Show that, if $\alpha + \beta + \gamma = \pi$,
$$\tan (\beta + \gamma - \alpha) + \tan (\gamma + \alpha - \beta) + \tan (\alpha + \beta - \gamma)$$
$$= \tan (\beta + \gamma - \alpha) \tan (\gamma + \alpha - \beta) \tan (\alpha + \beta - \gamma).$$

25. If $A + B + C = \pi$, prove that

(i) $\sin A \cos B \cos C + \sin B \cos C \cos A$
$$+ \sin C \cos A \cos B = \sin A \sin B \sin C.$$

(ii) $\cos A \sin B \sin C + \cos B \sin C \sin A$
$$+ \cos C \sin A \sin B = 1 + \cos A \cos B \cos C.$$

(iii) $\sin 5A + \sin 5B + \sin 5C$
$$= 4 \cos \frac{5A}{2} \cos \frac{5B}{2} \cos \frac{5C}{2}.$$

(iv) $(\tan A + \tan B + \tan C)(\cot A + \cot B + \cot C)$
$$= 1 + \sec A \sec B \sec C.$$

26. If $\cos A + \cos B + \cos C = 0$, show that
$$\cos 3A + \cos 3B + \cos 3C = 12 \cos A \cos B \cos C.$$
$$[\ Write \ \cos 3A = 4 \cos^3 A - 3 \cos A, \ etc. \ ]$$

27. If $x + y + z = \frac{1}{2}\pi$, prove that
$$\cos (x - y - z) + \cos (y - z - x) + \cos (z - x - y)$$
$$- 4 \cos x \cos y \cos z = 0.$$

28. Show that
$$\sin (y - z) + \sin (z - x) + \sin (x - y)$$
$$+ 4 \sin \frac{y - z}{2} \sin \frac{z - x}{2} \sin \frac{x - y}{2} = 0.$$

29. If $x + y + z = 0$, show that
$$\cot (z + x - y) \cot (x + y - z) + \cot (x + y - z) \ \cot (y + z - x)$$
$$+ \cot (y + z - x) \cot (z + x - y) = 1.$$

30. If $x + y + z = xyz$, prove that
$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}.$$

TRIGONOMETRICAL EQUATIONS AND
GENERAL VALUES

57. It will be apparent from Chapter IV that there are infinitely many angles, the trigonometrical ratios of which have a given value. For example, if $\sin \theta = \frac{1}{2}$, one value of θ (the smallest positive value) is known to be $30°$. Now, sines of supplementary angles are equal. Hence, $\sin 150°$ being equal to $\sin 30°$ is also $\frac{1}{2}$. Again, angles differing from $30°$ or $150°$ by complete multiples of $360°$ will have their sines (in fact all ratios) the same. Thus, sine of each of the angles $30°$, $150°$, $390°$, $510°$, $- 330°$, $- 210°$, etc. is equal to $\frac{1}{2}$. Similarly, if $\cos \theta$ be given, equal to $\frac{1}{\sqrt{2}}$ say, θ may have any of the values $+ 45°$, $+ 315°$, $+ 405°$, $- 315°$, $- 45°$, etc. ; or else, if $\tan \theta = \sqrt{3}$, θ may have any of the values $60°$, $240°$, $420°$, $- 300°$, etc.

It is very convenient for the solution of trigonometrical equations, as also for other purposes, to obtain a general expression in a compact form embracing all angles, the trigonometrical ratios of which have a given value.

58. General expression of all angles, one of whose trigonometrical ratios is zero.

, If the sine of an angle be zero, from definition, the length of the perpendicular from any point of one of its arms upon another is zero, so that the two arms must be in the same straight line. Evidently, therefore, such angles must be zero, or some multiple of π, odd or even.

Thus, if **$\sin \theta = 0$, then $\theta = n\pi$,**
n being zero, or any integer, positive or negative.

When the cosine of an angle is zero, the projection of any length along one arm upon another is zero, and so the two arms must be at right angles to one another. The angles must therefore be evidently either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or differ from these by complete revolutions ; in other words, the angle may be any odd multiple of $\frac{\pi}{2}$.

Thus, if $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$,

n being zero, or any integer, positive or negative.

Again, if $\tan\theta = 0$, then its numerator $\sin\theta$ is also zero ; and so $\theta = n\pi$.

Similarly, if $\cot\theta = 0$, then $\cos\theta = 0$;

and so $\theta = (2n+1)\frac{\pi}{2}$.

Note. The ratios $\operatorname{cosec}\theta$ or $\sec\theta$ can never be zero, for they can never be numerically less than unity.

59.   General expression of angles having the same sine *(or cosecant).*

Let a be any angle positive or negative such that its sine is equal to a given quantity k (numerically not greater than 1) ; for fixing up the idea, and for the sake of convenience in practice, the smallest positive angle having its sine for the given quantity k is taken as a. Let θ be any other angle whose sine is equal to k.

Then, $\sin\theta = \sin a$,

   or, $\sin\theta - \sin a = 0$,

   or, $2\sin\frac{1}{2}(\theta - a)\cos\frac{1}{2}(\theta + a) = 0$.

$\therefore$ either $\sin\frac{1}{2}(\theta - a) = 0$,

i.e., $\frac{1}{2}(\theta - a) =$ any multiple of $\pi = m\pi$, $\cdots$ (1)

or,   else $\cos \frac{1}{2}(\theta + a) = 0$,

i.e., $\frac{1}{2}(\theta + a) =$ any odd multiple of $\frac{\pi}{2} = (2m+1)\frac{\pi}{2}$. $\cdots$ (2)

From (1). $\theta - a = 2m\pi$,   *i.e.*, $\theta = a + 2m\pi$. $\cdots$ (3)

From (2), $\theta + a = (2m+1)\pi$, *i.e.*, $\theta = -a + (2m+1)\pi$ $\cdots$ (4)

Combining (3) and (4), $\theta = (-1)^n a + n\pi$ $\cdots$ (5)

where n is zero, or any integer, positive or negative, odd or even.

If $\operatorname{cosec} \theta = \operatorname{cosec} a$, then $\sin \theta = \sin a$; hence all angles having the same cosecant as that of a are also given by the expression (5).

Thus, *all angles having the same sine or cosecant as that of a are given by* $2n\pi + a$ *and* $(2n+1)\pi - a$,

or, $n\pi + (-1)^n a$.

60. General expression of angles having the same cosine (*or secant*).

Let a be the smallest positive angle such that its cosine is equal to a given quantity k (numerically $\not> 1$) ; and let θ be any other angle whose cosine is equal to k.

Then, $\cos \theta = \cos a$,

or, $\cos a - \cos \theta = 0$,

$\therefore$ $2 \sin \frac{1}{2}(\theta + a) \sin \frac{1}{2}(\theta - a) = 0$.

$\therefore$ either $\sin \frac{1}{2}(\theta + a) = 0$,

i.e., $\frac{1}{2}(\theta + a) =$ any multiple of $\pi = n\pi$ $\cdots$ (1)

or else, $\sin \frac{1}{2}(\theta - a) = 0$,

    i.e., $\frac{1}{2}(\theta - a) =$ any multiple of $\pi = n\pi$. $\cdots$ (2)

From (1), $\theta + a = 2n\pi$,   or, $\theta = 2n\pi - a$. $\cdots$ (3)

From (2), $\theta - a = 2n\pi$,   or, $\theta = 2n\pi + a$. $\cdots$ (4)

From (3) and (4), we have $\theta = 2n\pi \pm a$, $\cdots$ (5)

where n is zero, or any integer, positive or negative.

It is also evident as in the previous case that all angles having the same secant as that of a are also included in the expression (5).

Hence, *all angles having the same cosine or secant as that of a are given by*

$$2n\pi \pm a,$$

n being zero, or any integer, positive or negative.

Note. As in Art. 59, instead of taking the smallest positive angle, we might take a to be any one angle having for its cosine the given quantity k. The general values of θ satisfying $\cos \theta = \cos a$ as obtained above, would not be affected at all.

61. General expression of all angles having the same tangent *(or cotangent)*.

Let a be the smallest positive angle such that its tangent is equal to a given quantity k; and let θ be any other angle whose tangent is equal to k.

Then, $\tan \theta = \tan a$,

or, $\dfrac{\sin \theta}{\cos \theta} - \dfrac{\sin a}{\cos a} = 0$,

or, $\dfrac{\sin \theta \cos a - \cos \theta \sin a}{\cos \theta \cos a} = 0$,

or, $\dfrac{\sin (\theta - a)}{\cos \theta \cos a} = 0$.

$\therefore$ $\sin (\theta - a) = 0$,

i.e., $\theta - a =$ any multiple of $\pi = n\pi$.

$\therefore$ $\theta = a + n\pi$. $\cdots$ (1)

The factor $\dfrac{1}{\cos \theta \cos a}$ cannot be zero, for cosine of an angle cannot have an infinitely large value.

It is also evident as in the previous case that all angles having the same cotangent as that of a are given by the expression (1).

Hence, *all angles having the same tangent or cotangent as that of a are given by*

$$n\pi + a,$$

n being zero, or any integer, positive or negative.

Note. The remark below Art. 60 is applicable here also.

62.   Special cases.

From Art. 59, considering both cases when n is odd or even, it may be easily seen that

if $\sin \theta = 1 = \sin \dfrac{\pi}{2}$, $\theta = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}$;

and if $\sin \theta = -1 = \sin\left(-\dfrac{\pi}{2}\right)$, $\theta = 2n\pi - \dfrac{\pi}{2} = (4n-1)\dfrac{\pi}{2}$

$$\text{or,} \qquad = (4k+3)\dfrac{\pi}{2},$$

where n (or $k = n-1$) is zero, or any integer, positive or negative.

Similarly, from Art. 60, it may be seen that

          if $\cos \theta = 1$, $\theta = 2n\pi$

and   if $\cos \theta = -1$, $\theta = (2n+1)\pi$,

n being zero, or any integer, positive or negative.

These are the usual forms in which the above special cases are used in practice.

63. Geometrical Treatment.

(*i*) *Geometrical construction of an angle whose sine (or cosecant) is given, and to obtain a general expression of all such angles.*
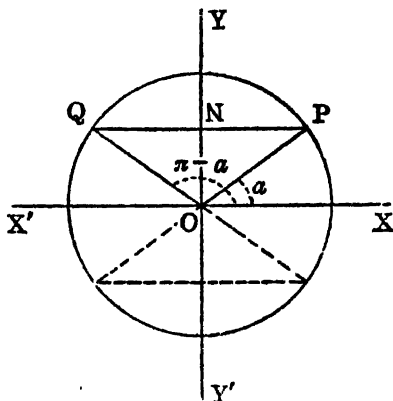
Let the sine of an angle be given equal to '*a*'.

Taking the perpendicular lines XOX' and YOY' for reference, draw a circle of unit radius with centre O.

Measure off $ON = a$ along OY (or along OY' if a be negative). Through N draw a straight line PNQ parallel to XOX' meeting the circle at P and Q.

Then, $\angle POX = a$ say, is one of the required angles, for

$$\sin a = \sin OPN \cdot \frac{ON}{OP} = \quad \cdot a.$$

Another angle with the same sine, as is apparent from the figure, is $\angle QOX = \pi - a$ (or $3\pi - a$ if $a = ON$ be negative, which is trigonometrically the same as $\pi - a$).

'*a*' being given in magnitude and sign, the position of N on YOY' is fixed and thus in one revolution, *i.e.* from 0 to

2π there are, as is clear from the figure, only two angles a and $\pi - a$ having the given sine.[*]

Now, the addition or subtraction of any multiple of 2π makes no difference in the values of the trigonometrical ratios of an angle (See *Art. 28*).

Hence, all the angles having the same sine as that of a are contained in the formulæ $2m\pi + a$ and $2m\pi + \pi - a$ i.e., $(2m + 1)\pi - a$, where m is zero, or any integer, positive or negative. Both the sets of angles are evidently included in the formula $n\pi + (-1)^n a$, n being zero, or any integer, positive or negative.

(ii) Angles with given cosine (or secant).
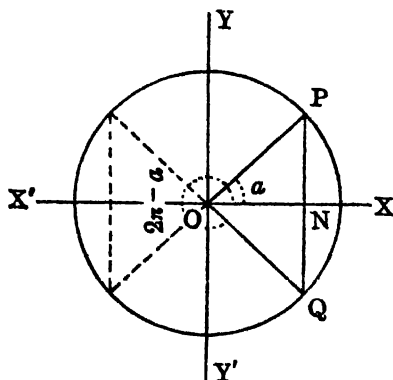
Let the given cosine be 'a'. As before, measure off $ON = a$ along OX (or along OX' if 'a' be negative), and through N draw PNQ parallel to YOY' to meet the circle with centre O and radius unity, at P and Q.

---

[*] In the same quadrant there cannot be two distinct angles (without being coterminals) having the same sine, for the corresponding triangles will then be congruent.

Let $\angle POX = a$. Then, a is a required angle. Also from the figure, the only angles in the first four quadrants which have the given cosine are a and $2\pi - a$.

Adding or subtracting multiples of 2π to these, all the angles having the same cosine as that of a are given by $2m\pi + a$ or $2m\pi + 2\pi - a$, both of which are included in the formula $2n\pi \pm a$, n being zero, or any integer, positive or negative.

(*iii*) *Angles with given tangent (or cotangent).*
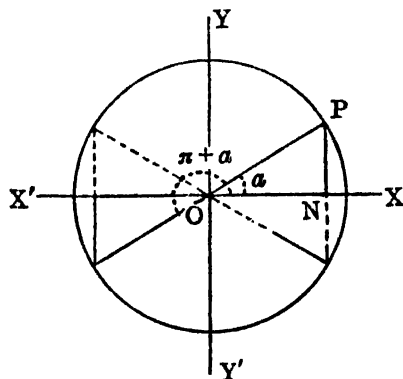
Let 'a' be the given tangent. Along OX or OX' measure off ON of unit length, and then measure off NP perpendicular to it of length whose numerical value is 'a'. If 'a' be positive, both ON and NP will be positive, or both will be negative, and so the $\angle XOP$ will be either in the first or in the third quadrant. If 'a' be negative, the angle will be either in the second or in the fourth quadrant. In any case there are only two angles, within one revolution, *i.e.*, from 0 to 2π as is apparent from the figure, with the given tangent.[*]

---

[*]The ratio $PN : ON$ being given, and the included angle PNO being right, the triangle PNO constructed remains always similar to itself and so in the same quadrant $\angle PON$ of the triangle is unique.

One of the angles being a, the other is evidently (from the figure) $\pi + a$. Adding or subtracting multiples of 2π, all the angles having the same tangent as that of a are given by $2m\pi + a$ or $2m\pi + \pi + a$ both of which are included in the formula $n\pi + a$ where n is zero, or any integer, positive or negative, odd or even.

Ex. 1.   *Solve* $2\,(cos^2\theta - sin^2\theta) = 1$.

The given equation can be written as

$\qquad$ $2\cos 2\theta = 1$. $\qquad\qquad$ $\therefore$ $\quad \cos 2\theta = \tfrac{1}{2} = \cos \tfrac{1}{3}\pi$.

$\therefore$ $\quad 2\theta = 2n\pi \pm \tfrac{1}{3}\pi$. $\qquad\qquad$ $\therefore$ $\quad \theta = n\pi \pm \tfrac{1}{6}\pi$.

Note. It may be observed that a trigonometrical equation can be solved in several ways ; and the results though different in forms will give the same series of angles. To illustrate this we work out the above example in another way.

The equation can also be written in the form

$\qquad$ $2(\cos^2\theta - 1 + \cos^2\theta) = 1$, or, $4\cos^2\theta = 3$.

$\therefore$ $\quad \cos\theta = \pm\, \dfrac{\sqrt{3}}{2} = \cos\dfrac{\pi}{6}$, or, $\cos\dfrac{5\pi}{6}$.

$\therefore$ $\quad \theta = 2m\pi \pm \dfrac{\pi}{6}$, or, $2m\pi \pm \dfrac{5\pi}{6}$.

$\qquad$ Now, $2m\pi \pm \dfrac{5\pi}{6} = (2m+1)\pi - \dfrac{\pi}{6}$, or, $(2m-1)\pi + \dfrac{\pi}{6}$.

All the four sets of solutions, m being any integer, can be included in the expression $n\pi \pm \tfrac{1}{6}\pi$, in which form the result has already been obtained by the previous process.

Ex. 2.   Solve $4\,cos^2x + 6\,sin^2x = 5$.

The equation can be written as

$\qquad$ $4\cos^2x + 6\sin^2x = 5\,(\sin^2x + \cos^2x)$.

$\therefore$ $\quad \sin^2x = \cos^2x$, or, $\tan^2x = 1$.

$\therefore$ $\quad \tan x = \pm 1$. $\quad \therefore$ $\quad x = n\pi \pm \tfrac{1}{4}\pi$.

Note. Equations of the form $a\,cos^2x + b\,sin^2x = c$ can be easily solved by the above method, or by expressing sine in terms of cosine or cosine in terms of sine.

Ex. 3.   *Solve* $2\,sin^2x + sin^2\,2x = 2$. $\qquad$ [*C. U. 1940*]

The given equation can be written as

$\qquad$ $2(1 - \sin^2x) - \sin^2 2x = 0$, or, $2\cos^2x - 4\sin^2x\cos^2x = 0$.

or, $2 \cos^2 x \,(1 - 2 \sin^2 x) = 0$, or, $\cos^2 x \cos 2x = 0$.

∴ either $\cos x = 0$, *i.e.*, $x = n\pi + \tfrac{1}{2}\pi$,

or, $\cos 2x = 0$, *i.e.*, $2x = 2n\pi \pm \tfrac{1}{2}\pi$. ∴ $x = n\pi \pm \tfrac{1}{4}\pi$.

Ex. 4. *Solve* $\cos \theta - \sin \theta = \dfrac{1}{\sqrt{2}}$.

Dividing both sides of the equation by $\sqrt{1^2 + 1^2}$, *i.e.*, $\sqrt{2}$, we have

$$\cos \theta . \frac{1}{\sqrt{2}} - \sin \theta . \frac{1}{\sqrt{2}} = \frac{1}{2},$$

i.e., $\cos \theta \cos \tfrac{1}{4}\pi - \sin \theta \sin \tfrac{1}{4}\pi = \tfrac{1}{2}$.

∴ $\cos (\theta + \tfrac{1}{4}\pi) = \cos \tfrac{1}{3}\pi$. ∴ $\theta + \tfrac{1}{4}\pi = 2n\pi \pm \tfrac{1}{3}\pi$.

∴ $\theta = 2n\pi + \tfrac{1}{12}\pi$, or, $2n\pi - \tfrac{7}{12}\pi$.

Note. Extraneous solutions.

In general, as pointed out in Ex. 1 above, the same trigonometrical equation may be solved by different methods, and the forms of the result we arrive at, though apparently different in some cases, are ultimately equivalent. In some cases, however, we may be tempted to solve a trigonometrical equation by methods which have flaws in them leading to solutions which include in addition to the correct solutions, some extraneous solutions which do not satisfy the given equation. The given equation which is of the type $a \cos \theta + b \sin \theta = c$ is an example. We proceed to demonstrate it as follows :

Here, $\cos \theta - \dfrac{1}{\sqrt{2}} = \sin \theta$.

∴ $\cos^2 \theta - \sqrt{2} \cos \theta + \tfrac{1}{2} = \sin^2 \theta = 1 - \cos^2 \theta$,

whence $2 \cos^2 \theta - \sqrt{2} \cos \theta - \tfrac{1}{2} = 0$.

∴ $\cos \theta = \dfrac{\sqrt{2} \pm \sqrt{2 + 4}}{4} = \dfrac{1 \pm \sqrt{3}}{2\sqrt{2}} = \cos \dfrac{\pi}{12}$, or, $\cos \dfrac{7\pi}{12}$.

∴ $\theta = 2n\pi \pm \tfrac{1}{12}\pi$, or, $2n\pi \pm \tfrac{7}{12}\pi$.

But it can be easily seen on substitution that

$2n\pi - \tfrac{1}{12}\pi$ and $2n\pi + \tfrac{7}{12}\pi$ do not satisfy the given equation. *The error in the method lies in squaring the equation* as we have done ; for the squared equation includes the equation $\cos \theta - \dfrac{1}{\sqrt{2}} = - \sin \theta$,

i.e., $\cos \theta + \sin \theta = \dfrac{1}{\sqrt{2}}$ of which the solutions are $2n\pi - \tfrac{1}{12}\pi$ and $2n\pi + \tfrac{7}{12}\pi$.

Equations of this type are therefore best solved as in the next example, and not by squaring.

Thus, *while solving any trigonometrical equation it is always advisable to verify the roots obtained* ; for thereby extraneous roots, if any, can be easily detected.

Ex. 5. *Solve* $a \cos \theta + b \sin \theta = c$. $(c \ngtr \sqrt{a^2 + b^2})$

Put $a = r \cos a$, $b = r \sin a$, choosing the smallest positive value of a, keeping r positive.

Then, $r = \sqrt{a^2 + b^2}$ and $\sin a = \dfrac{b}{\sqrt{a^2 + b^2}}$

$$\text{and } \cos a = \dfrac{a}{\sqrt{a^2 + b^2}}.$$

The signs of a and b will determine the quadrant in which a lies, and a and b being given, r and a are definitely known.

The equation now becomes

$$r \cos (\theta - a) = c,$$

$$\text{or, } \cos (\theta - a) = \dfrac{c}{\sqrt{a^2 + b^2}} = \cos \beta,$$

where β is the smallest positive angle whose cosine is $\dfrac{c}{\sqrt{a^2 + b^2}}$, and a, b, c being known, β is also known.

Hence, $\theta - a = 2n\pi \pm \beta$,   or, $\theta = 2n\pi + a \pm \beta$.

Note. An angle which is introduced in a trigonometrical work to facilitate calculations is called a *subsidiary angle*. Thus, a and β are here subsidiary angles.

Ex. 6. *Solve* $4 \cos x + 5 \sin x = 5$, *given* $\tan 51° \, 21' = \frac{5}{4}$.

Dividing both sides of the given equation by $\sqrt{4^2 + 5^2}$, *i.e.*, by $\sqrt{41}$, we get

$$\frac{4}{\sqrt{41}} \cos x + \frac{5}{\sqrt{41}} \sin x = \frac{5}{\sqrt{41}}. \qquad \cdots \quad (1)$$

Since, $\tan 51° \, 21' = \frac{5}{4}$,

$$\therefore \quad \sin 51° \, 21' = \frac{5}{\sqrt{41}}, \ \cos 51° \, 21' = \frac{4}{\sqrt{41}}.$$

$\therefore$ (1) reduces to

$\cos x \cos 51° 21' + \sin x \sin 51° 21' = \sin 51° 21'$

or, $\cos (x - 51° 21') = \sin 51° 21' = \cos 38° 39'$.

$\therefore$ $x - 51° 21' = 2n\pi \pm 38° 39'$.

$\therefore$ $x = 2n\pi + 90°$, or, $2n\pi + 12° 42'$.

Ex. 7. (i) *Solve* $2 \sin^2 x + \sin^2 2x = 2$ *for* $-\pi < x < \pi$.

From Ex. 3 above, we see that $x = n\pi + \frac{1}{2}\pi$ $\cdots$ (1)

   or, $x = n\pi \pm \frac{1}{4}\pi$. $\cdots$ $\cdots$ $\cdots$ (2)

Putting $n = 0, -1$ in (1), we get $x = \frac{1}{2}\pi, -\frac{1}{2}\pi$, which lie in the given interval. Putting $n = 0, 1, -1$ in (2), we get $x = \pm \frac{1}{4}\pi, \frac{3}{4}\pi, -\frac{3}{4}\pi$ which also lie in the given interval.

Hence, the required values of x are $\pm \frac{1}{4}\pi, \pm \frac{1}{2}\pi, \pm \frac{3}{4}\pi$.

(ii) *Solve* $\cos \theta + \sqrt{3} \sin \theta = 2$

$\qquad$ *for* $-2\pi < \theta < 2\pi$ *and* $3\pi < \theta < 5\pi$.

Dividing both sides of the equation by $\sqrt{1+3}$, *i.e.*, 2, we have

$$\cos \theta \cdot \frac{1}{2} + \sin \theta \cdot \frac{\sqrt{3}}{2} = 1,$$

i.e., $\cos \theta \cdot \cos \frac{1}{3}\pi + \sin \theta \cdot \sin \frac{1}{3}\pi = 1$,

i.e., $\cos (\theta - \frac{1}{3}\pi) = 1$.

$\therefore$ $\theta - \frac{1}{3}\pi = 2n\pi$. *i.e.*, $\theta = 2n\pi + \frac{1}{3}\pi$.

Putting $n = 0, -1$, we get $\theta = \frac{1}{3}\pi, -\frac{5}{3}\pi$ which lie in the 1st interval.

Again, putting $n = 1, 2$, we get $\theta = \frac{7}{3}\pi, \frac{13}{3}\pi$, which lie in the 2nd interval.

Ex. 8.   *Solve* $\tan ax = \cot bx$.

Here, $\tan ax = \cot bx = \tan (\frac{1}{2}\pi - bx)$.

$\therefore$ $ax = n\pi + \frac{1}{2}\pi - bx$.

$\therefore$ $x = \frac{2n+1}{a+b} \cdot \frac{\pi}{2}$.

Examples XI

Solve the following equations (*Ex. 1 to 23*) :—

1. $\cot^2 x + \operatorname{cosec}^2 x = 3$.

2. (i) $2 \cos^2 \theta + 4 \sin^2 \theta = 3$.

(ii) $\tan^2 \theta = 3 \operatorname{cosec}^2 \theta - 1$.   [*C. U. 1939*]

3. $\tan x - \cot x = \operatorname{cosec} x$.

4. $\cot x - \cot 2x = 2$.

5. $2 \sin \theta \tan \theta + 1 = \tan \theta + 2 \sin \theta$.

6 $\sin 5\theta + \sin \theta = \sin 3\theta$.

7. $\sin m\theta + \sin n\theta = 0$.

8. $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$.

9. $\cot 2x = \cos x + \sin x$.

10. $\sin x + \cos x = \sqrt{2}$, for $-\pi < x < \pi$.

11. $\sin 2x \tan x + 1 = \sin 2x + \tan x$.

12. $\cot x - \tan x = 2$.   [*C. U. 1934, '37*]

13. $\sin x + \sqrt{3} \cos x = \sqrt{2}$.   [*C. U. 1938, '47*]

14. $2 \sin x \sin 3x = 1$.

15. $\sin \theta + 2 \cos \theta = 1$.   [*C. U. 1933*]

16. $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$.

17. $\tan (\tfrac{1}{4}\pi + \theta) + \tan (\tfrac{1}{4}\pi - \theta) = 4$.   [*C. U. 1949*]

18. $\tan x + \tan 2x + \tan x \tan 2x = 1$.   [*C. U. 1941, '45*]

19. $\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}$.   [*C. U. 1944*]

20. $\sqrt{3} \cos x + \sin x = 1$, for $-2\pi < x < 2\pi$.

21. $\cos 2x = \cos x \sin x$.

22. $2 \cot x + \sin x = 2 \operatorname{cosec} x$.

23. $\cos x + \sin x = \cos 2x + \sin 2x$.   [*C. U. 1943*]

24. Solve $2 \sin^2 x + \sin x = 3$; and find all the angles between 0° and 1000° which satisfy it.

25. Find the solution of the equations (general solution is not required)

$$\tan x + \tan y = 2$$
$$2 \cos x \cos y = 1.$$

26. If $\tan ax - \tan bx = 0$, show that the values of x form a series in A.P.

27. Solve

(i) $\cos 3x + \cos 2x + \cos x = 0$.   [*C. U. 1941, '49*]

(ii) $\cos 9x \cos 7x = \cos 5x \cos 3x$, $-\frac{1}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$.

(iii) $\tan x + \tan 2x + \tan 3x = 0$.   [*A. I. 1941*]

(iv) $\cos x - \sin x = \cos a + \sin a$.   [*B. H. U. 1938*]

(v) $\cos^3 x - \cos x \sin x - \sin^3 x = 1$.

(vi) $\cos 6x + \cos 4x = \sin 3x + \sin x$.

(vii) $\dfrac{\sin a}{\sin 2x} + \dfrac{\cos a}{\cos 2x} = 2$.

28. Salve $5 \cos \theta + 2 \sin \theta = 2$, given $\tan 68° \, 12' = 2\frac{1}{2}$.

29. Find those pairs of solutions of the following equations which correspond to positive solutions less than 2π of each individual equation :—

(i) $\sin (a - \beta) = 0$; $\sin (a + \beta) = 1$.

(ii) $\sin (a - \beta) = \cos (a + \beta) = \frac{1}{2}$.

30. If $\sin A = \sin B$, $\cos A = \cos B$, prove that either A and B are equal or they differ by some multiple of four right angles.   [*C. U. 1935*]

31. Show that the three equations

$$\sin^2 \theta = \sin^2 a, \ \cos^2 \theta = \cos^2 a, \ \tan^2 \theta = \tan^2 a$$

are all identical and the solution is always $n\pi \pm a$.

32. Show that the same two series of angles are given by the equations

$$x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{6} \text{ and } x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

INVERSE CIRCULAR FUNCTIONS

64. The equation $\sin \theta = x$ means that θ is an angle whose sine is x. It is often convenient to express this statement *inversely* by writing $\theta = \sin^{-1}x$. Thus, the symbol $\sin^{-1}x$ denotes an angle whose sine is x. Hence, $\sin^{-1}x$ is an angle, whereas $\sin \theta$ is a number. The two relations $\sin \theta = x$ and $\theta = \sin^{-1}x$ are identical ; if one is given the other follows. The symbol $\sin^{-1}x$ is usually read as "*sine inverse x*". Sometimes it is also denoted by *arc sin x*.

Note. $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$, *i.e.*, $\dfrac{1}{\sin x}$.

65. We know that if θ be any one angle whose sine is equal to x, then sines of all the angles given by $n\pi + (-1)^n\theta$ are equal to x. Hence, $\sin^{-1}x$ has got an infinite number of values, and as such, $\sin^{-1}x$ is a *multiple-valued function*.

Hence, the *general value of* $sin^{-1}x = n\pi + (-1)^n \, sin^{-1}x$ where on the rigt-hand side $\sin^{-1}x$ stands for any particular angle whose sine is x.

Similarly, the *general value of*
$$cos^{-1}x = 2n\pi \pm cos^{-1}x$$
and of $tan^{-1}x = n\pi + tan^{-1}x$.

The smallest numerical value, either positive or negative, of θ is called the *principal value* of $\sin^{-1}x$. Thus, the principal value of $\sin^{-1}\frac{1}{2}$ is $30°$. If corresponding to the same ratio, there are two numerically equal angles, one positive and the other negative, it is customary to take the positive angle as the principal value ; thus, the principal value of $\cos^{-1}\frac{1}{2}$ is $60°$, and not $(-60°)$ although $\cos(-60°) = \frac{1}{2}$.

In all numerical examples, the principal value is generally taken.

$\cos^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ have similar significance and all properties as those of $\sin^{-1}x$. These expressions are called **Inverse Circular Functions.**

66. If $\sin\theta = x$, then $\theta = \sin^{-1}x$, *i.e.*, $\theta = \sin^{-1}\sin\theta$.

Similarly, $\theta = \cos^{-1}\cos\theta = \tan^{-1}\tan\theta$; etc.

Again, if $\theta = \sin^{-1}x$, $\sin\theta = x$, *i.e.*, $\sin\sin^{-1}x = x$.

Similarly, $\cos\cos^{-1}x = x$; $\tan\tan^{-1}x = x$; etc.

Also, we have

$$\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x} ; \cot^{-1}x = \tan^{-1}\frac{1}{x} ; \sec^{-1}x = \cos^{-1}\frac{1}{x}.$$

Let $\text{cosec}^{-1}x = \theta$; then $\text{cosec}\,\theta = x$.

$$\therefore\quad \sin\theta = \frac{1}{\text{cosec}\,\theta} = \frac{1}{x}.$$

Hence, $\theta = \sin^{-1}\frac{1}{x}$, and therefore, $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}.$

In the same way we have, $\text{cosec}^{-1}\frac{1}{x} = \sin^{-1}x.$

The other relations follow similarly.

67. As all the trigonometrical ratios can be expressed in terms of any one, similarly all the inverse trigonometrical ratios can be expressed in terms of any one inverse ratio.

Thus, let $\sin^{-1}x = \theta$; then $\sin\theta = x$,

$$\therefore\quad \cos\theta = \sqrt{1-x^2} ; \tan\theta = \frac{x}{\sqrt{1-x^2}} ; \cot\theta = \frac{\sqrt{1-x^2}}{x} ;$$

$$\sec\theta = \frac{1}{\sqrt{1-x^2}} \text{ and } \text{cosec}\,\theta = \frac{1}{x}.$$

$$\therefore\quad \theta = \sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}.$$

68.   *To prove that*

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$.

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}$.

(i) Let $\sin^{-1}x = \theta$; then $\sin\theta = x$.

Now, $\sin\theta = \cos\left(\tfrac{1}{2}\pi - \theta\right)$.

$\therefore$ $\cos\left(\tfrac{1}{2}\pi - \theta\right) = x$ and hence $\cos^{-1}x = \tfrac{1}{2}\pi - \theta$.

Therefore, $\sin^{-1}x + \cos^{-1}x = \theta + \tfrac{1}{2}\pi - \theta = \tfrac{1}{2}\pi$.

(ii) Let $\tan^{-1}x = \theta$; then $\tan\theta = x$.

Now, $\tan\theta = \cot\left(\tfrac{1}{2}\pi - \theta\right)$.

$\therefore$ $\cot\left(\tfrac{1}{2}\pi - \theta\right) = x$. $\therefore$ $\cot^{-1}x = \tfrac{1}{2}\pi - \theta$.

$\therefore$ $\tan^{-1}x + \cot^{-1}x = \theta + \tfrac{1}{2}\pi - \theta = \tfrac{1}{2}\pi$.

(iii) Let $\operatorname{cosec}^{-1}x = \theta$; then $\operatorname{cosec}\theta = x$.

Now, $\operatorname{cosec}\theta = \sec\left(\tfrac{1}{2}\pi - \theta\right)$.

$\therefore$ $\sec\left(\tfrac{1}{2}\pi - \theta\right) = x$. $\therefore$ $\sec^{-1}x = \tfrac{1}{2}\pi - \theta$.

$\therefore$ $\operatorname{cosec}^{-1}x + \sec^{-1}x = \theta + \tfrac{1}{2}\pi - \theta = \tfrac{1}{2}\pi$.

69.   *To prove that*

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$.

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$.

Let $\tan^{-1}x = \alpha$; and $\tan^{-1}y = \beta$;

then $\tan\alpha = x$; and $\tan\beta = y$.

Now, $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\,\tan\beta} = \dfrac{x+y}{1-xy}$.

$$\therefore \quad a+\beta = \tan^{-1}\frac{x+y}{1-xy},$$

i.e., $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}.$

Again, $\tan(a-\beta) = \dfrac{\tan a - \tan \beta}{1+\tan a \tan \beta} = \dfrac{x-y}{1+xy}.$

$$\therefore \quad a-\beta = \tan^{-1}\frac{x-y}{1+xy},$$

i.e., $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}.$

Note. It can be easily proved as above that

$$\cot^{-1}x \pm \cot^{-1}y = \cot^{-1}\frac{xy \mp 1}{y \pm x}.$$

70.   *To prove that*

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

Let $\tan^{-1}x = a$; $\tan^{-1}y = \beta$; $\tan^{-1}z = \gamma.$

$\therefore \quad \tan a = x,\ \ \tan \beta = y,\ \ \tan \gamma = z.$

Now, $\tan(a+\beta+\gamma)$

$$= \frac{\tan a + \tan \beta + \tan \gamma - \tan a \tan \beta \tan \gamma}{1-\tan \beta \tan \gamma - \tan \gamma \tan a - \tan a \tan \beta}$$

$$= \frac{x+y+z-xyz}{1-yz-zx-xy}.$$

Hence, $a+\beta+\gamma = \tan^{-1}\dfrac{x+y+z-xyz}{1-yz-zx-xy}.$

Since, $a+\beta+\gamma = \tan^{-1}x + \tan^{-1}y + \tan^{-1}z$, the required result follows.

Note. This relation can also be deduced by applying twice the formula of Art. 69. Thus,

Left side $=(\tan^{-1}x + \tan^{-1}y) + \tan^{-1}z$

$= \tan^{-1}\dfrac{x+y}{1-xy} + \tan^{-1}z$; now again apply Art. 69.

71. In fact, for most of the formulæ involving ordinary circular functions, corresponding relations connecting the inverse circular functions can be easily deduced. In addition to those given above, some are illustrated in the following examples :

Ex. 1. *Show that*

(i) $sin^{-1}x \pm sin^{-1}y = sin^{-1} \{x \sqrt{1-y^2} \pm y \sqrt{1-x^2}\}.$

(ii) $cos^{-1}x \pm cos^{-1}y = cos^{-1} (xy \mp \sqrt{(1-x^2)(1-y^2)}\}.$

(i) Let $sin^{-1}x = a.$ $\therefore$ $sin\, a = x$ and $cos\, a = \sqrt{1-x^2}$;

also let $sin^{-1}y = \beta.$ $\therefore$ $sin\, \beta = y$ and $cos\, \beta = \sqrt{1-y^2}.$

Now, $sin\,(a \pm \beta) = sin\, a\, cos\, \beta \pm cos\, a\, sin\, \beta$
$$= x \sqrt{1-y^2} \pm y \sqrt{1-x^2}.$$
$$\therefore \quad a \pm \beta = sin^{-1} \{x \sqrt{1-y^2} \pm y \sqrt{1-x^2}\}.$$

Since, $a \pm \beta = sin^{-1}x \pm sin^{-1}y$, the required result follows.

(ii) These relations follow similarly from the value of $cos\,(a \pm \beta)$.

Ex. 2. *Show that*

(i) $2\, sin^{-1}x = sin^{-1} (2x \sqrt{1-x^2}).$

(ii) $2\, cos^{-1}x = cos^{-1} (2x^2 - 1).$

(iii) $2\, tan^{-1}x = tan^{-1} \dfrac{2x}{1-x^2}.$

(i) Let $sin^{-1}x = a.$ $\therefore$ $sin\, a = x$, $cos\, a = \sqrt{1-x^2}.$

Now, $sin\, 2a = 2\, sin\, a\, cos\, a = 2x \sqrt{1-x^2}.$

$\therefore \qquad 2a = sin^{-1} (2x \sqrt{1-x^2}).$

Since, $a = sin^{-1}x$, the required result follows.

(ii) & (iii). These relations follow similarly from the corresponding values of $cos\, 2a$ in terms of $cos\, a$ and of $tan\, 2a$ in terms of $tan\, a$. [See *Art. 43*]

Note. The above three relations can also be deduced by putting x for y in the values of $\sin^{-1}x + \sin^{-1}y$, $\cos^{-1}x + \cos^{-1}y$ and $\tan^{-1}x + \tan^{-}y$.

Ex. 3. *Show that*

(i) $3 \sin^{-1}x = \sin^{-1}(3x - 4x^3)$.

(ii) $3 \cos^{-1}x = \cos^{-1}(4x^3 - 3x)$.

(iii) $3 \tan^{-1}x = \tan^{-1}\dfrac{3x - x^3}{1 - 3x^2}$.   [*C. U. 1938*]

(i) Let $\sin^{-1}x = \theta$; then $\sin \theta = x$.

Now, $\sin 3\theta = 3 \sin \theta - 4 \sin^3\theta = 3x - 4x^3$.

∴ 3θ, *i.e.*, $3 \sin^{-1}x = \sin^{-1}(3x - 4x^3)$.

(ii) & (iii).   These relations follow similarly from the corresponding values of $\cos 3\theta$ in terms of $\cos \theta$ and of $\tan 3\theta$ in terms of $\tan \theta$.   [See *Art. 44*]

Note. The result (iii) may also be deduced by putting $y = z = x$ in the formula of Art. 70.

Ex. 4. *Show that*

$$2 \tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}.$$

Let $\tan^{-1}x = \theta$,   ∴ $\tan \theta = x$.

Since, $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2\theta} = \dfrac{2x}{1 + x^2}$,   [*Art. 45, Ex. 1*]

∴ 2θ, *i.e.*, $2 \tan^{-1}x = \sin^{-1}\dfrac{2x}{1 + x^2}$.

Since, $\cos 2\theta = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \dfrac{1 - x^2}{1 + x^2}$,

and $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2\theta} = \dfrac{2x}{1 - x^2}$,

the remaining relations follow similarly.

Ex. 5. *Show that*

$$tan^{-1} \frac{a-b}{1+ab} + tan^{-1} \frac{b-c}{1+bc} + tan^{-1} \frac{c-a}{1+ca} = 0.$$

1st term of left side $= tan^{-1}a - tan^{-1}b$ [*By Art. 69 (ii)*],

2nd $\cdots$ $\cdots$ $\cdots$ $= tan^{-1}b - tan^{-1}c$,

3rd $\cdots$ $\cdots$ $\cdots$ $= tan^{-1}c - tan^{-1}a$.

Hence, adding up the three terms, the required result follows.

Ex. 6. *Show that*

$$2\ tan^{-1} \tfrac{1}{5} + tan^{-1} \tfrac{1}{4} = tan^{-1} \tfrac{32}{43}.$$

Since, $2 \tan^{-1}x = \tan^{-1} \dfrac{2x}{1-x^2}$,   [See *Ex. 4*]

$$\therefore\ 2 \tan^{-1} \tfrac{1}{5} = \tan^{-1} \frac{\frac{2}{5}}{1 - \frac{1}{5^2}} = \tan^{-1} \tfrac{5}{12}.$$

$\therefore$ left side $= \tan^{-1} \tfrac{5}{12} + \tan^{-1} \tfrac{1}{4} = \tan^{-1} \dfrac{\tfrac{5}{12} + \tfrac{1}{4}}{1 - \tfrac{5}{48}} = \tan^{-1} \tfrac{32}{43}.$

Ex. 7. *Solve*

$$sin^{-1} \frac{2a}{1+a^2} + sin^{-1} \frac{2b}{1+b^2} = 2\ tan^{-1}x.$$

$$[\ C.\ U.\ 1947\ ]$$

Since, $\sin^{-1} \dfrac{2x}{1+x^2} = 2 \tan^{-1}x$,   [See *Ex. 4*]

$\therefore$ left side $= 2 \tan^{-1}a + 2 \tan^{-1}b$.

$\therefore$ the equation reduces to

$$2 \tan^{-1}x = 2 \tan^{-1}a + 2 \tan^{-1}b.$$

$\therefore$ $\tan^{-1}x = \tan^{-1}a + \tan^{-1}b = \tan^{-1} \dfrac{a+b}{1-ab}.$

$$x = \frac{a+b}{1-ab}.$$

Ex. 8. *Solve*

$$tan^{-1} \frac{x-1}{x-2} + tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}.$$

Left side $= tan^{-1} \dfrac{\dfrac{x-1}{x-2} + \dfrac{x+1}{x+2}}{1 - \dfrac{x^2-1}{x^2-4}} = tan^{-1} \dfrac{2x^2-4}{-3}.$

$\therefore$ the equation reduces to

$$tan^{-1} \frac{2x^2-4}{-3} = \frac{\pi}{4} = tan^{-1} 1.$$

$\therefore$ $\dfrac{2x^2-4}{-3} = 1$ or, $2x^2 = 1$ or, $x = \pm \dfrac{1}{\sqrt{2}}.$

Examples XII

Prove that (*Ex. 1 to 17*):—

1. (i) $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{1}{4}\pi.$

   (ii) $\tan^{-1} x + \tan^{-1} \dfrac{2x}{1-x^2} = \tan^{-1} \dfrac{3x-x^3}{1-3x^2}.$

   (iii) $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} = \cot^{-1} 3.$

2. $\tan^{-1} \frac{2}{11} + \cot^{-1} \frac{24}{7} = \tan^{-1} \frac{1}{2}.$

3. $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$
   $= 2 (\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}).$

4. (i) $\tan^{-1} x + \cot^{-1} (x+1) = \tan^{-1} (x^2+x+1).$

   (ii) $\tan^{-1} \dfrac{1}{p+q} + \tan^{-1} \dfrac{q}{p^2+pq+1} = \tan^{-1} \dfrac{1}{p}.$

5. $\tan^{-1} a - \tan^{-1} c = \tan^{-1} \dfrac{a-b}{1+ab} + \tan^{-1} \dfrac{b-c}{1+bc}.$

6. $\tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{27}{11}.$

7. $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{1}{4}\pi.$

   [*C. U. 1942*]

8. $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{1}{4}\pi.$ [*C. U. 1937*]

9. (i) $\sin (2 \sin^{-1} x) = 2x \sqrt{1 - x^2}$.

(ii) $\{\cos (\sin^{-1} x)\}^2 = \{\sin (\cos^{-1} x)\}^2$.

10. $\cos^{-1} x = 2 \sin^{-1} \sqrt{\dfrac{1 - x}{2}} = 2 \cos^{-1} \sqrt{\dfrac{1 + x}{2}}$.

11. $\tan^{-1} \sqrt{x} = \dfrac{1}{2} \cos^{-1} \dfrac{1 - x}{1 + x}$.   [*C. U. 1943*]

12. $\sin^{-1} \sqrt{\dfrac{x - b}{a - b}} = \cos^{-1} \sqrt{\dfrac{a - x}{a - b}} = \tan^{-1} \sqrt{\dfrac{x - b}{a - x}}$.

13. $\tan^{-1} \dfrac{a - b}{1 + ab} + \tan^{-1} \dfrac{b - c}{1 + bc} + \tan^{-1} \dfrac{c - a}{1 + ca}$

$= \tan^{-1} \dfrac{a^2 - b^2}{1 + a^2 b^2} + \tan^{-1} \dfrac{b^2 - c^2}{1 + b^2 c^2} + \tan^{-1} \dfrac{c^2 - a^2}{1 + c^2 a^2}$

14. $\sec^2 (\tan^{-1} 2) + \operatorname{cosec}^2 (\cot^{-1} 3) = 15$.

15. $\cot^{-1} (\tan 2x) + \cot^{-1} (- \tan 3x) = x$.

16. $\sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{16}{65} = \frac{1}{2}\pi$.   [*C. U. 1941*]

17. $4 (\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5}) = \pi$.   [*C. U. 1939*]

18.   If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, show that
$$x + y + z = xyz.$$

19.   If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{1}{2}\pi$, show that
$$yz + zx + xy = 1.$$

20.   If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, show that
$$x^2 + y^2 + z^2 + 2xyz = 1.$$

21.   If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, show that
$$x \sqrt{1 - x^2} + y \sqrt{1 - y^2} + z \sqrt{1 - z^2} = 2xyz.$$

22.   Find the values of

(i) $\sin (\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{2})$.   [*C. U. 1935*]

(ii) $\cot (\tan^{-1} a + \cot^{-1} a)$.

(iii) $\tan \left(\frac{1}{2} \sin^{-1} \dfrac{2x}{1 + x^2} + \frac{1}{2} \cos^{-1} \dfrac{1 - y^2}{1 + y^2} \right)$.

23. If $\tan^{-1}y = 4\tan^{-1}x$, find y as an algebraic function of x.

24. If $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P., find out the algebraic relation between x, y, z. If in addition, x, y, z are also in A.P., prove that $x = y = z$.   [$y \neq 0, 1$ or -1]

25. Solve the following equations :

    (i) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$.

    (ii) $\tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$.

    (iii) $\tan(\cos^{-1}x) = \sin(\tan^{-1}2)$.

    (iv) $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x$.

    (v) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$.

    (vi) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.

    (vii) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$.

    (viii) $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$.

    (ix) $\tan^{-1}\frac{2x}{1-x^2} + \cot^{-1}\frac{1-x^2}{2x} = \frac{\pi}{3}$.

    (x) $\cot^{-1}(x-1) + \cot^{-1}(x-2) + \cot^{-1}(x-3) = 0$.

26. Show that

    (i) $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x} = 0$.

    (ii) $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z)$
                   $= \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$.

    (iii) $\tan^{-1}(\cot x) + \cot^{-1}(\tan x) = \pi - 2x$.

Miscellaneous Examples I

1. If $3 \sin \theta + 4 \cos \theta = 5$, show that $\tan \theta = \frac{3}{4}$.

2. If $a^2 \sec^2 x - b^2 \tan^2 x = c^2$, find cosec x.

3. If $x = r \cos \theta \cos \phi$, $y = r \cos \theta \sin \phi$, $z = r \sin \theta$, show that $x^2 + y^2 + z^2 = r^2$.

4. If $\sin \theta = \dfrac{x-y}{x+y}$, show that $\tan \left(\dfrac{\pi}{4} - \dfrac{\theta}{2} \right) = \pm \sqrt{\dfrac{y}{x}}$.

5. If $x = r \sin (\theta + 45°)$ and $y = r \sin (\theta - 45°)$, then $x^2 + y^2 = r^2$.

6. If $\cos (\alpha + \beta) \sin (\gamma + \theta) = \cos (\alpha - \beta) \sin (\gamma - \theta)$, then $\tan \theta = \tan \alpha \tan \beta \tan \gamma$.

Show that (*Ex. 7 to 9*) :—

7. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \dfrac{x-y}{2}$.

8. $\sin A + \sin B + \sin C - \sin (A + B + C)$
$$= 4 \sin \frac{A+B}{2} \sin \frac{B+C}{2} \sin \frac{C+A}{2}.$$

9. $4 \sin \dfrac{A+B+C}{2} \sin \dfrac{B+C-A}{2} \sin \dfrac{C+A-B}{2} \sin \dfrac{A+B-C}{2}$
$$= 1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C.$$

10. If $\tan \beta = \dfrac{2 \sin \alpha \sin \gamma}{\sin (\alpha + \gamma)}$, then $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are in harmonical progression.

11. If $\alpha + \beta + \gamma = (2n + 1) \dfrac{\pi}{2}$, then

(i) $\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1$.

(ii) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = \pm 4 \cos \alpha \cos \beta \cos \gamma$.

12. If the angles A, B, C be in A.P., then
$$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{\cos B}{\sin B}.$$

13. If $\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0$, show that
$$\tan A + \tan B + \tan C + \cot A + \cot B + \cot C = 0.$$

14. If $\tan \alpha = \dfrac{a \sin \beta}{1 - a \cos \beta}$ and $\tan \beta = \dfrac{b \sin \alpha}{1 - b \cos \alpha}$,

then $\qquad \dfrac{\sin \alpha}{\sin \beta} = \dfrac{a}{b}$.

15. Show that
$$\tan \theta + 2 \tan 2\theta + 4 \tan 4\theta + 8 \cot 8\theta = \cot \theta.$$

16. If $\cos (\theta - \psi) \cos \phi = \cos (\theta - \phi + \psi)$, then $\tan \theta$, $\tan \phi$, $\tan \psi$ are in harmonical progression.

17. If $1 + \cos (y - z) + \cos (z - x) + \cos (x - y) = 0$, show that either $(y - z)$, or $(z - x)$, or $(x - y)$ is an odd multiple of π.

18. If $\sin \theta + \sin \phi = \sqrt{3} (\cos \phi - \cos \theta)$, show that
$$\sin 3\theta + \sin 3\phi = 0.$$

19. Eliminate α and β from
$$\sin \alpha + \sin \beta = a,\ \cos \alpha + \cos \beta = b,\ \cos (\alpha - \beta) = c.$$

20. If $A + B + C = \pi$, prove that

(i) $\tan B \tan C + \tan C \tan A + \tan A \tan B$
$$= 1 + \sec A \sec B \sec C.$$

(ii) $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
$$+ \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

21. If $A + B + C = \pi$, and if
$$\sin^2 A + \sin^2 B + \sin^2 C = \sin B \sin C + \sin C \sin A$$
$$+ \sin A \sin B, \text{ then } A = B = C.$$

22. If A, B, C be the angles of a triangle, and if
$\cot A + \cot B + \cot C = \sqrt{3}$, show that the triangle is equilateral.

23. If $\sec ax + \sec bx = 0$, show that the values of x form two series in A.P.

72. Definition of Logarithm.

Logarithm of a number with respect to a given base is the index of the power to which the base is to be raised in order to give the number.

Mathematically, if $a^x = N$, then 'x' is the index of the power to which 'a' (which is called the base) is raised to give 'N'. Hence, by definition, 'x' is the logarithm of 'N' with respect to the base 'a' and it is usually written as $x = \log_a N$.

As a numerical example, $\log_2 8 = 3$, for $2^3 = 8$ *i.e.*, 3 is the power to which 2 is to be raised to give 8. Again, since $3^4 = 81$, $4 = \log_3 81$.

Any result involving indices can be expressed as a result in logarithm, and *vice versa*.

For example,

$$\text{if } p^q = r, \quad \text{then, } q = \log_p r,$$
$$\text{if } m^n = z^k, \text{ then } n = \log_m (z^k),$$
$$\text{or, } k = \log_z (m^n).$$

Similarly, if $\log_y x = z,$

$$\text{then} \quad y^z = x.$$

It should be noted that the logarithms of the same number with respect to different bases will be different ; for example, to get the same number 64, we must raise 2 to the power 6, whereas we are to raise 4 to the power 3 and 8 to the power 2 only ; hence $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_8 64 = 2$.

Thus, so long as the base is not stated, logarithm of a number has no meaning.

73. Special Cases.

We know from Algebra that if a be any real finite quantity, other than zero, then $a^0 = 1$.

Hence, $\log_a 1 = 0$; in other words.

(*i*) *logarithm of 1 with respect to any finite quantity (other than zero) as base, is zero.*

Again, a being any quantity, $a^1 = a$.

Hence, $1 = \log_a a$; in other words,

(*ii*) *logarithm of any number with respect to itself as base is unity.*

Note 1. If $a^x = 0$, then $x = -\infty$ if $a > 1$, and $x = +\infty$ if $a < 1$.

Thus, we have $\log_a 0 = \mp \infty$ according as $a >$ or < 1. Hence, *logarithm of zero to a base greater than unity is minus infinity, and to a base less than unity is plus infinity.*

Note 2 Since the equation $a^x = -n$ (a and n being real positive quantities), cannot be satisfied by any real value of x, whether positive or negative, provided we consider the principal value* only of a^x, therefore, *logarithm of a negative quantity* (in a system of logarithms whose base is a real positive quantity) *must be imaginary.*

74. Fundamental formulæ in logarithms.

From the definition it is clear that logarithms are but indices in another form. Hence, corresponding to the three fundamental results in the theory of indices in Algebra, namely that if a, x, y be any real quantities,

(i) $a^x \times a^y = a^{x+y}$,

(ii) $a^x \div a^y = a^{x-y}$, and

(iii) $(a^x)^y = a^{xy}$,

we get three fundamental laws of logarithms which are given below.

---

* See a treatise on Higher Trigonometry.

(i) $\log_a (m \times n) = \log_a m + \log_a n$.

In other words, *logarithm of the product of two quantities is equal to the sum of their logarithms taken separately, base remaining the same always.*

Proof. Put $\log_a m = x$, $\log_a n = y$
and $\log_a (m \times n) = z$.

Then from definition,
$$a^x = m,\ a^y = n \text{ and } a^z = m \times n = a^x \times a^y = a^{x+y},$$

so that, $z = x + y$.

Replacing the values,
$$\log_a (mn) = \log_a m + \log_a n.$$

Cor. $\log_a (m.n.p......) = \log_a m + \log_a n + \log_a p + \cdots\cdots$

(ii) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$.

In other words, *logarithm of the quotient of two numbers is equal to the difference of their logarithms* (logarithm of the numerator minus logarithm of the denominator).

Proof. Put $\log_a m = x$, $\log_a n = y$
and $\log_a \left(\dfrac{m}{n}\right) = z$.

Then, from definition,
$$a^x = m,\ a^y = n$$
and $$a^z = \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

so that
$$z = x - y,$$

or replacing the values,
$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

(iii) $\log_a (m)^n = n \log_a m.$

Or, *logarithm of a power of a number is the product of the power and the logarithm of the number.*

Proof.   Put $\log_a m = x$, and $\log_a (m)^n = z$.

Then, by definition,

$$a^x = m \quad \text{and}$$
$$a^z = (m)^n = a^{nx}.$$
$$\therefore \quad z = nx,$$

or replacing the values,

$$\log_a (m)^n = n \log_a m.$$

Ex. 1.   *Reduce to a simple form* $\log_a \dfrac{x^p y^q}{z^s}.$

$$\log_a \frac{x^p y^q}{z^s} = \log_a (x^p y^q) - \log_a (z^s)$$
$$= \log_a x^p + \log_a y^q - \log_a z^s$$
$$= p \log_a x + q \log_a y - s \log_a z.$$

Ex. 2.   *Simplify* $\log_{10} \sqrt[3]{\frac{25}{88}}.$

$$\log_{10} \sqrt[3]{\frac{25}{88}} = \log_{10} \left(\frac{5^2}{8.11} \right)^{\frac{1}{3}} = \tfrac{1}{3} \log_{10} \frac{5^2}{2^3.11}$$
$$= \tfrac{1}{3} \log_{10} \frac{10^2}{2^5.11}$$
$$= \tfrac{1}{3} \left[\log_{10} 10^2 - \log_{10} (2^5.11) \right]$$
$$= \tfrac{1}{3} \left[2 \log_{10} 10 - (\log_{10} 2^5 + \log_{10} 11) \right]$$
$$= \tfrac{1}{3} \left[2 - 5 \log_{10} 2 - \log_{10} 11 \right].$$

75.   Change of base.

There is a fourth standard formula whereby logarithms of numbers with respect to one base being given, those with respect to a different base may be obtained. The formula is

$$\log_a m = \log_b m \times \log_a b.$$

Proof.   Put $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.
Then, from definition,

$$a^x = m,\ b^y = m,\ a^z = b.$$

Hence, $x^x = m = b^y = (a^z)^y = a^{yz}$,

   or, $x = yz$.

Replacing the values,

$$\log_a m = \log_b m \times \log_a b.$$

Cor. 1.   In the above result, put $m = a$.   Then remembering that $\log_a a = 1$, we get

$$\log_b a \times \log_a b = 1.$$

Since, the above relation is very important, we add here an *idependent proof* of it.

Let $\log_b a = x$, and $\log_a b = y$.

Then, $b^x = a$ and $a^y = b$.

$\therefore$ $a = b^x = (a^y)^x = a^{xy}$. $\therefore$ $xy = 1$,

i.e., $\log_b a \times \log_a b = 1$,

or, $\log_b a = \dfrac{1}{\log_a b}$.

Cor. 2.   The result of the above article may be written with the help of Cor. 1, in the form

$$\log_a m = \log_b m / \log_b a.$$

Thus, if logarithms of both m and a with respect to b be known, logarithm of m with respect to a is obtained.

76.   Common system of logarithms.

For all practical purposes wherever logarithms are used for numerical calculations, the base is usually taken as 10.   Logarithms of numbers with respect to the base 10 are

referred to as the *Common system* of logarithms. The
advantage of the common system of logarithms for practical
applications will be clear presently, from the Article 77,
Theorems I & II.

Note. In higher mathematics, for *theoretical* investigations,
another quantity '*e*' (defined in books of Algebra), whose value is nearly
2·718... is used as the base of logarithms, and logarithms to this base
e are called *Napierian* logarithms.

With the help of the logarithmic series established in books
on Algebra, Napierian logarithms of numbers are tabulated. The
factor $\dfrac{1}{\log_e 10}$ which is known as the *modulus of the common system*,
applied to the Napierian logarithms will convert them to common
logarithms (See *Art. 75*). Thus, a table of common logarithms is
prepared.

Henceforth, we shall proceed with the consideration of
the common system of logarithms, and the base being
understood to be 10, will not be written.

77. Characteristic and Mantissa *of common logarithms.*

It is only in very few cases that the logarithm of
a number is integral. In most cases, however, the logar-
ithm of a number is partly integral and partly fractional
(or decimal).

Def. The integral portion of the logarithm of a number
is called the *characteristic*, and the decimal portion is called
the *mantissa*.

In case the logarithm of a number is negative, and
partly integral and partly decimal, the decimal portion, *i.e.*,
the mantissa is always kept positive by altering the integral
part, *i.e.*, the characteristic suitably. Thus, the *mantissa
part of the logarithm of a number is always positive.* For
instance, if the logarithm of a number is $-2·3$, we write
it as $-3 + ·7$ and call -3 as the characteristic and ·7 (and
not $-·3$) as the mantissa. $-3 + ·7$ is often abbreviated in
the form $\bar{3}·7$.

Theorem I. *The* **characteristic** *of the common logarithm of (i) any number greater than 1 is positive, and numerically one less than the number of digits in the integral part of the quantity whose logarithm is sought ; and (ii) of any positive[*] number less than 1, is negative, and numerically one greater than the number of zeros immediately after the decimal point in the quantity whose logarithm is wanted.*

(i) Let the number be greater than unity.

Any number, say 7˙209, which consists of 1 digit only in its integral part, lies between 1 and 10.

Now, $10^0 = 1$ and $10^1 = 10$.

Hence, if $10^x = 7˙209$, clearly x must be greater than 0 and less than 1.

Thus, log 7˙209 must be between 0 and 1, *i.e.*, of the form 0˙..., having its characteristic 0.

Similarly, numbers of the type 53˙0528, which consists of 2 digits in their integral parts, must lie between 10 and 100 *i.e.*, between 10^1 and 10^2.

Hence, the index to which 10 should be raised to give 53˙0528 must be greater than 1 and less than 2, *i.e.*, log 53˙0528 must be of the form 1˙... having the characteristic 1.

log 10 is 1, and 10 also falls in this category of two digits.

In the same way, a number which has n digits in its integral part lies between 10^{n-1} (which also has n *digits*) and 10^n (which has $n+1$ *digits*). Thus, the logarithms of such numbers must lie between $n-1$ and n, *i.e.*, $(n-1)$ + some positive proper fraction. Hence, the characteristic in such cases is $n-1$.

Hence, the result.

---

[*]Logarithms of negative numbers are easily seen to be imaginary, for there is no real power, positive or negative, to which 10 may be raised to give a negative result. [See *Note 2, Art. 73*]

(ii) Let .the number be positive, and less than 1 (*i.e.*, between 0 and 1).

We notice that

$$10^0 \qquad\qquad = 1$$

$$10^{-1} = \frac{1}{10} \qquad\quad = \dot{}1$$

$$10^{-2} = \frac{1}{100} \qquad\quad = \dot{}01$$

$$10^{-3} = \frac{1}{1000} \qquad = \dot{}001$$

$$10^{-4} = \frac{1}{10000} \qquad = \dot{}0001$$

etc.   etc.   etc.

Now, a number less than 1, with no zero immediately after the decimal point, like ˙3015, must be greater than ˙1 and less than 1 ; hence, the power to which 10 must be raised to give such a number must lie between -1 and 0, *i.e.*, $= -1 +$ a positive proper fraction. Hence, such numbers have the characteristic of their logarithms $= -1$.

A decimal number with one zero immediately after the decimal point, like ˙078005, lies between ˙01 and ˙1 which are respectively equal to 10^{-2} and 10^{-1}.

Hence, if $10^x = \dot{}078005$, x must lie between -1 and -2 *i.e.*, x is of the form $-1\cdot\cdots\cdots$ Writing the decimal part of x positively, in the form $-2 + \cdots\cdots$, we notice that the integral part of x, *i.e.*, the characteristic of the logarithm of ˙078005 is -2.

Similarly, the logarithms of numbers between ˙01 and ˙001 (*i.e.*, 10^{-2} and 10^{-3}) which must have two zeros after the decimal point, lie between -2 and -3, *i.e.*, are of the form $-2\cdot\cdots\cdots = -3 + \cdots\cdots$, and so the characteristic in such cases is -3,

and so on.

Hence the result.

Theorem II. *All numbers, formed of the same digits in the same order, differing only in the positions of their decimal points, have the* **mantissæ** *of their logarithms same.*

This will be clear from an example. Let us take the numbers 835107, 835107000, 83·5107, ·835107, ·000835107 and 8351·07.

$$
\begin{aligned}
\text{Now,} \quad \log 835107000 &= \log (835107 \times 1000) \\
&= \log 835107 + \log 1000 \\
&= \log 835107 + \text{·}3.
\end{aligned}
$$

$$
\begin{aligned}
\text{Again,} \quad \log 83\text{·}5107 \quad &= \log \frac{835107}{10000} \\
&= \log 835107 - \log 10000 \\
&= \log 835107 - 4.
\end{aligned}
$$

$$
\log \text{·}835107 \quad = \log \frac{835107}{1000000} = \log 835107 - 6.
$$

$$
\log \text{·}000835107 = \log \frac{835107}{10^9} = \log 835107 - 9.
$$

$$
\log 8351\text{·}07 \quad = \log \frac{835107}{100} = \log 835107 - 2.
$$

Thus, the logarithms of all the numbers here differ from the logarithm of 835107 by a whole number in each case and so must have their decimal parts, *i.e.*, their mantissæ the same as that of log 835107.

In fact, numbers formed of the same digits in the same order differing only in the position of their decimal points, must have their ratios equal to an integral power of 10 and so must have their logarithms differing only by a whole number.

Hence the result.

The two theorems above given show that (i) the characteristic of the logarithm of a number can be found by a simple glance at the number and (ii) that for the mantissa part of the logarithm of a number, we need only take into

account the digits of which the number is formed, without taking any notice of the position of the decimal point in it.

In logarithmic tables, only the mantissæ of the logarithms of numbers are therefore given.

These constitute the special advantages of the common system of logarithms.

78. Examples worked out.

Ex. 1. *Simplify*

$$\log \frac{\sqrt[4]{5}.\sqrt[10]{2}}{\sqrt[3]{(18.\sqrt{2})}}, \text{ and find its value, given}$$

$$\log 2 = {}^{\cdot}30103 \text{ and } \log 3 = {}^{\cdot}4771213.$$

The given exp. $= \log \dfrac{5^{\frac{1}{4}}.2^{\frac{1}{10}}}{(18.2^{\frac{1}{2}})^{\frac{1}{3}}}$

$$-\log \frac{10^{\frac{1}{4}}.2^{\frac{1}{10}}}{2^{\frac{1}{4}}(2.3^2.2^{\frac{1}{2}})^{\frac{1}{3}}} = \log \frac{10^{\frac{1}{4}}.2^{\frac{1}{10}}}{2^{\frac{1}{4}}.2^{\frac{1}{3}}.3^{\frac{2}{3}}.2^{\frac{1}{6}}}$$

$$= \log \frac{10^{\frac{1}{4}}}{2^{\frac{13}{20}}.3^{\frac{2}{3}}} = \log 10^{\frac{1}{4}} - \log (2^{\frac{13}{20}} \times 3^{\frac{2}{3}})$$

$$= \tfrac{1}{4} \log 10 - (\log 2^{\frac{13}{20}} + \log 3^{\frac{2}{3}})$$

$$= \tfrac{1}{4} \log 10 - \tfrac{13}{20} \log 2 - \tfrac{2}{3} \log 3$$

and its value is

$$\tfrac{1}{4}.1 - \tfrac{13}{20}({}^{\cdot}30103) - \tfrac{2}{3}({}^{\cdot}4771213)$$

$$= {}^{\cdot}25 - {}^{\cdot}1956695 - {}^{\cdot}3180809$$

$$= -1 + {}^{\cdot}7362496$$

$$= \bar{1}{}^{\cdot}7362496.$$

Note. $\log 5 = \log \tfrac{10}{2} = \log 10 - \log 2 = 1 - \log 2$ and hence $\log 5$ is deducible from $\log 2$.

Ex. 2. *Prove that*

$$7 \log \tfrac{10}{9} - 2 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80} = \log 2.$$

The left-hand expression

$$= \log \left(\tfrac{10}{9}\right)^7 - \log \left(\tfrac{25}{24}\right)^2 + \log \left(\tfrac{81}{80}\right)^3$$

$$= \log \frac{\left(\tfrac{10}{9}\right)^7 \times \left(\tfrac{81}{80}\right)^3}{\left(\tfrac{25}{24}\right)^2}$$

$$= \log \left\{ \left(\frac{10}{3^2}\right)^7 \times \left(\frac{3^4}{10 \times 2^3}\right)^3 \times \left(\frac{3 \times 2^3 \times 2^2}{10^2}\right)^2 \right\}$$

$$= \log \left(\frac{10^7}{3^{14}} \times \frac{3^{12}}{10^3 \times 2^9} \times \frac{3^2 \times 2^{10}}{10^4} \right)$$

$$= \log 2.$$

Alternative method :

Left side

$$= 7(\log 10 - \log 9) - 2(\log 25 - \log 24) + 3(\log 81 - \log 80)$$

$$= \{\log (5 \times 2) - \log 3^2\} - 2\{\log 5^2 - \log (3 \times 2^3)\}$$
$$+ 3 \{\log 3^4 - \log (5 \times 2^4)\}$$

$$= 7\{\log 5 + \log 2 - 2 \log 3\} - 2\{2 \log 5 - \log 3 - 3 \log 2\}$$
$$+ 3\{4 \log 3 - \log 5 - 4 \log 2\}$$

$$= \log 2.$$

Ex. 3. *Find the number of digits in* 4^{15}, *having given* $\log 2 = \cdot30103$.

We have

$$\log 4^{15} = \log 2^{30} = 30 \log 2$$
$$= 30 \times \cdot30103 = 9\cdot0309.$$

Hence, since the characteristic of $\log 4^{15}$ is 9, 4^{15} must consist of 10 digits.

Ex. 4. *Find approximately the* 7th *root of* $3\cdot528$, *having given* $\log 2 = \cdot30103$, $\log 3 = \cdot4771213$. $\log 7 = \cdot8450980$ *and* $\log 1197\cdot342 = 3\cdot0782184$.

Let $x = (3\cdot528)^{\frac{1}{7}} = \left(\frac{7^2 \times 3^2 \times 2^3}{10^3}\right)^{\frac{1}{7}}$

then $\log x = \tfrac{1}{7} [2 \log 7 + 2 \log 3 + 3 \log 2 - 3 \log 10]$

$$= \tfrac{1}{7} [2 \times \cdot8450980 + 2 \times \cdot4771213 + 3 \times \cdot30103 - 3]$$

$$= \cdot0782184 \text{ nearly.}$$

Now, $\log 1197\cdot342 = 3\cdot0782184$.

∴ $\log 1\cdot197342 = \cdot0782184$, having characteristic 0, but mantissa same as that of $\log 1197\cdot342$.

Hence, $x = 1\cdot197342$ approximately.

Ex. 5.   *Obtain an approximate numerical solution of* $2^x.3^{2x} = 100$, *having given log* $2 = \cdot30103$, *log* $3 = \cdot47712$.

We have

$$2^x.3^{2x} = 10^2.$$

$$\therefore \quad \log(2^x.3^{2x}) = \log 10^2,$$

i.e., $x \log 2 + 2x \log 3 = 2 \log 10 = 2.$

$$\therefore \quad x = \frac{2}{\log 2 + 2 \log 3} = \frac{2}{\cdot30103 + 2 \times \cdot47712}$$

$$= 1\cdot5933 \text{ nearly.}$$

Note.   Equations of this type are called **Exponential Equations.**

Examples XIII(a)

[*Use the values* :   *log* $2 = \cdot30103$,   *log* $3 = \cdot4771213$,
       log $7 = \cdot8450980$ *when required*]

1.   Find the logarithm of (i) 1728 to the base $2\sqrt{3}$, (ii) $\cos^3 a$ to the base sec a.

2.   Find $\log_{\cdot01} 10000$.

3.   Show that $\log_{10} 2$ lies between $\frac{1}{3}$ and $\frac{1}{4}$.

[*C. U. 1926*]

4.   Prove that

   (i) $\log_a m \times \log_b n = \log_b m \times \log_a n.$

   (ii) $\log_2 \log_2 \log_2 16 = 1.$

5.   If $\log_e m + \log_e n = \log_e(m+n)$, find m as a simple function of n.

6.   Prove that if a series of numbers be in G.P., their logarithms are in A.P.

7. Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 + \cdots\cdots + 2 \log a^n$$
$$= n (n + 1) \log a.$$

8. If x is positive and less than unity, show that
$$\log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8) + \cdots \text{ to } \infty$$
$$= - \log (1 - x).$$

9. Simplify

(i) $\log_2 \sqrt{6} + \log_2 \sqrt{\tfrac{2}{3}}$.

(ii) $\dfrac{\log \sqrt{27} + \log 8 - \log \sqrt{1000}}{\log 1 \cdot 2}$.

10. Find $\log (\cdot 00225)^{\frac{1}{3}}$ and $\log (\tfrac{5}{72})^{-\frac{1}{3}}$.

11. Prove that

(i) $\log_a b \times \log_b c \times \log_c a = 1$.

(ii) $\log_a x = \log_b x \times \log_c b \times \log_d c \cdots \times \log_n m \times \log_a n$.

12. Show that

(i) $7 \log \tfrac{10}{12} + 5 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80} = \log 2$.

(ii) $7 \log \tfrac{15}{16} + 6 \log \tfrac{8}{3} + 5 \log \tfrac{2}{5} + \log \tfrac{32}{25} = \log 3$.

13. Extract the fifth root of 84, having given
$$\log 2425805 = 6 \cdot 3848559.$$

14. Calculate $(\cdot 0020736)^{\frac{1}{7}}$, having given
$$\log 41369 = 4 \cdot 6166750.$$

15. Simplify

(i) $\log \sqrt[7]{\dfrac{8^{\frac{1}{5}} \times 14^{\frac{1}{6}}}{\sqrt{72} \times \sqrt[5]{60}}}$.

(ii) $\sqrt[3]{\dfrac{7 \cdot 2 \times 6 \cdot 3}{62 \cdot 5}}$, having given
$$\log 898665 = 5 \cdot 9535977.$$

16. Find the value of $64\{1-(1\cdot05)^{-20}\}$, having given

$$\log 24121 = 4\cdot382394.$$

17. Find the number of digits in (i) 2^{40}, (ii) 3^{11}, (iii) $(540)^9$.

18. Find the number of zeros after the decimal point before the first significant digit in the expressions

(i) $(\cdot024)^{15}$.　　　(ii) $\left(\dfrac{1}{4\cdot05}\right)^8$.　　　(iii) $\left(\cdot0259\right)^{50}$.

19. Solve the equations :—

(i) $3^x = 2$.　　　　　　　(ii) $3^{x-4} = 7$.

(iii) $5^{6x} \div 7^{x+2} = 3^{2x-3}$.

(iv) $\left.\begin{array}{l} 2^x = 3^y \\ 2^{y+1} = 3^{x-1} \end{array}\right\}$　　　(v) $\left.\begin{array}{l} 7^{x+y} \times 3^{2x+y} = 9 \\ 3^{x-y} \div 2^{x-2y} = 3^x \end{array}\right\}$

20. (i) If $\log(x^2y^3) = a$, $\log\left(\dfrac{x}{y}\right) = b$, find $\log x$ and $\log y$.

(ii) If $a^2 + b^2 = 7ab$, show that

$$\log\{\tfrac{1}{3}(a+b)\} = \tfrac{1}{2}(\log a + \log b).$$

21. If $\dfrac{\log x}{y-z} = \dfrac{\log y}{z-x} = \dfrac{\log z}{x-y}$, show that $x^x y^y z^z = 1$.

22. Why is $\log(1+2+3) = \log 1 + \log 2 + \log 3$?

23. If $a, b, c, \ldots\ldots$ be in G.P., show that

$$\log_a x, \log_b x, \log_c x \ldots\ldots \text{ are in H.P.}$$

24. If $xy^{l-1} = a$, $xy^{m-1} = b$, $xy^{n-1} = c$, prove that

$$(m-n)\log a + (n-l)\log b + (l-m)\log c = 0.$$

25. If $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z}$, show that

$$y^z z^y = z^x x^z = x^y y^x.$$

79. Tables of Logarithms and Trigonometrical ratios.

Several mathematical tables correct up to five places of decimals are given at the end of the book. An explanation of the tables is given below.

Table I gives the common logarithm of all numbers from 1 to 10000, *i.e.*, those which consist of 4 digits or less. The tabulated quantities are the mantissæ only, correct to five places, with the decimal point dropped. The characteristic is to be supplied according to the rule given in Art. 77. The main body of the table gives logarithms (mantissa part) of numbers of 3 digits, and the mean difference table at the side supplies the increment in the mantissa due to the fourth digit. This increment is written, in order to save space, giving the significant digits only, which are to be supplied with the necessary number of zeros to make up 5 places (here the table being a five-figure table). Thus, ˙00024 will be written as 24 only in the difference table. As an example, to find log 2˙697, we notice from the table that the mantissa for log 269 is ˙42975, and along the same row, the difference table gives 115 under the heading 7. This means that for 7 in the fourth place of the number (*i.e.*, for the number 2697) the increment in the mantissa will be ˙00115. Hence, log 2697 will have its mantissa ˙42975 + ˙00115 = ˙43090. Again, log 2˙697 has the same mantissa but its characteristic is 0. Thus, log 2˙697 = 0˙43090.

Table II gives ordinary sines and cosines (usually referred to as *natural sines and cosines*) of all angles from 0° to 90° at intervals of 1′, sines being given from the left side of the top towards the right and downwards, and cosines being given from the right side of the bottom towards the left and upwards. The table is arranged in such a way that the sine of any angle given is the same as the cosine of exactly the complementary angle, and it is on this arrangement that a single table serves as a sine as well as a cosine table. The main portion of the table gives sines or cosines of angles at intervals of 10′, and the difference

table at the side gives changes in the value of the sine or cosine for changes in minutes in the angles. It should be remembered that as an angle increases from 0° to 90°, its sine increases from 0 to 1 whereas its cosine decreases from 1 to 0. Hence, *the changes given in the difference table are to be added in case of sines and subtracted in case of cosines* for the increased number of minutes in the angles. Moreover, as in Table I, the numbers in the difference table are to be made up to five places of decimals by supplying the requisite number of zeros before it. For example, using the table, sin 53° 23′ = ˙80212 + ˙00052 = ˙80264 and cos 20° 42′ = ˙86892 − ˙00029 = ˙86863.

Table III similarly gives natural tangents and cotangents of angles from 0° to 90°, obtained at intervals of 1′ with the help of the difference table. The quantities in the difference table, being made up into five figures, are to be *added in case of tangents and subtracted in case of cotangents* for increased number of minutes in the angle.

Table IV gives logarithmic sines and logarithmic cosines of all angles from 0° to 90° at intervals of 1′ (with the aid of the difference table). Logarithmic sine of angle θ, written as L sin θ means $10 + \log \sin \theta$, and similarly, logarithmic cosine of θ, written as L cos θ means $10 + \log \cos \theta$. In taking logarithms of trigonometrical ratios of angles, it may be noted that sines and cosines of angles are numerically less than unity, and tangents of angles between 0° and 45° as also cotangents of angles between 45° and 90° are less than unity. Hence, logarithms of these quantities are negative. To avoid using negative values in the tables, *logarithms of trigonometrical ratios are always tabulated after adding* 10 *to them.* Thus, the table gives L sin θ and L cos θ (and not log sin θ and log cos θ).

Table V gives logarithmic tangents (*i.e.*, L tan $\theta = 10 + \log \tan \theta$) and logarithmic cotangents (*i.e.*, L cot $\theta = 10 + \log \cot \theta$) of all angles from 0° to 90°, obtained at intervals of 1′ with the aid of the difference table.

80. Principle of Proportional Parts.

Suppose we find from table I the logarithms of the two numbers 6257 and 6258, and we want to find the logarithm of 6257'6 ; or that we find from table III, tan 53° 23' and tan 53° 24', but we want to find tan 53° 23' 20''; or similarly, from table IV, we get L cos 37° 42' and L cos 37° 43' but we want to find L cos 37° 42' 48''; how are we to proceed ?

In order to meet such cases, the 'Principle of Proportional Parts' may be used. The principle may be stated as follows :

If the value of a quantity depending on a variable quantity x be tabulated for different values of x at regular small intervals, then in most cases, for a very small change in x (which is called the argument) the corresponding small change in the tabulated quantity (called the function of the argument) is proportional to the change in x.

We shall assume the truth of this principle ; for a strict proof of it, with the proper restriction under which it is true, depends on the use of Calculus. For the tables with which we are concerned, it is true for all practical purposes.

The application of the principle is illustrated in the following examples :

Ex. 1. *Given log 63374 = 4˙8019111 and log 63375 = 4˙8019180, find log 63˙3743 and find the number whose logarithm is $\overline{2}$˙8019136.*

Here,   log 63375 = 4˙8019180

and   log 63374 = 4˙8019111.

Hence, for an increase of 1 in the number, the increment in the logarithm is ˙0000069. (This is usually spoken as 'diff. for 1 is 69')

Therefore, by the Principle of Proportional Parts, increase in the logarithm for an increase of ˙3 in the number is

$$˙3 × ˙0000069 = ˙00000207$$
$$= ˙0000021, \text{ up to seven places.}$$

Hence, log 63374˙3 = 4˙8019111 + ˙0000021
$$= 4˙8019132.$$

∴   log 63˙3743 = 1˙8019132.

Again, 4˙8019136 lies between 4˙8019111 and 4˙8019180, the difference from the former being ˙0000025. Hence, 4˙8019136 is the logarithm of a number lying between 63374 and 63375, say logarithm of 63374 + x.

Then, diff. for 1 being 69 (*i.e.*, ˙0000069) and diff. for x being 25, (*i.e.*, ˙0000025), by the Principle of Proportional Parts, we have

$$69 : 25 :: 1 : x$$
$$\text{or,} \quad x = \tfrac{25}{69} = ˙36\cdots\cdots$$

Hence, log 63374˙36⋯ = 4˙8019136.

The required number whose logarithm is $\bar{2}$˙8019136, having the same mantissa, must be formed of the same digits arranged in the same order, and its characteristic being − 2, the number must be ˙06337436⋯

Ex. 2. (i) *Given L sin* 37° 43′ 50″ = 9˙7867152
$$L\ sin\ 37°\ 44′\quad = 9˙7867424,$$
find L sin 37° 43′ 56″.

(ii) *Given L tan* 79° 51′ 40″ = 10˙7475657
$$L\ tan\ 79°\ 51′\ 50″ = 10˙7476872,$$
find the angle whose L tan is 10˙7476532.

[*C. U. 1921*]

In (i) diff. (in the value of L sin) for $10''$ (diff. in angle)
$$= 272 \ (i.e., \ 0000272)$$
hence, diff. for $6'' = \frac{6}{10} \times 272 = 163 \cdot 2 \ i.e., \ 00001632$

and so L sin $37° \ 43' \ 56'' = 9 \cdot 7867152 + 0000163$
$$= 9 \cdot 7867315.$$

In (ii) the angle whose L tan is $10 \cdot 7476532$ evidently lies between $79° \ 51' \ 40''$ and $79° \ 51' \ 50''$.

Let the angle be $79° \ 51' \ 40'' + x''$.

Now, diff. (in the value of L tan) for $10''$ (diff. in angle)
$$= 1215 \ (i.e., \ 0001215)$$
and diff. for $x'' = 875$

$(i.e., \ 0000875, \text{ being } 10 \cdot 7476532 - 10 \cdot 7475657)$

$\therefore \quad \dfrac{x}{10} = \dfrac{875}{1215}$ or $x = 7 \cdot 2$ nearly.

Thus, the required angle is $79° \ 51' \ 47''·2$.

Ex. 3. *Given* $\cos 58° \ 17' = 5257191$ *and diff. for* $1' = 2474$, *find* $\cos 58° \ 17' \ 20''$.

Here, diff. for $1'$ *i.e.*, $60'' = 2474$,

$\therefore \quad$ diff. for $20'' = \frac{20}{60} \times 2474 = 825$ (nearly).

As for increasing angle, cosine diminishes,

$\therefore \quad \cos 58° \ 17' \ 20'' = 5257191 - 0000825$
$$= 5256366.$$

Examples XIII(b)

1. Given $\log 18 \cdot 906 = 1 \cdot 2765997$
and $\log 18 \cdot 907 = 1 \cdot 2766226$,
find $\log 1890 \cdot 635$.

2. Given $\log 69714 = 4 \cdot 8433200$
$\log 69715 = 4 \cdot 8433262$,
find $\log (000697145)^{\frac{1}{2}}$.

3.   Given log 37602 = 4·5752109

log 37601 = 4·5751994,

find the number whose logarithm is 1·5752086.

4.   Given log 3 = ·4771213

log 74008 = 4·8692787

diff. for $1'$ = 59,

find $(·09)^{\frac{1}{8}}$.

5.   Given cos 32° 16' = ·8455726

and   cos 32° 17' = ·8454172,

find the value of cos 32° 16' 24"

and find the angle whose cosine is ·8455176.

6.   Find tan 38° 24' 37·5", having given

tan 38° 24' = ·7925902 and tan 38° 25' = ·7930640.

7.   Given L sin 44° 17' = 9·8439842

and L sin 44° 18' = 9·8441137,

find L sin 44° 17' 33".   Deduce the value of

L cosec 44° 17' 33".

8.   Given L sin 36° 24' = 9·7733614

L sin 36° 25' = 9·7735327,

find the angle whose L sin is 9·7734642.

9.   If L cot 53° 13' = 9·8736937

L cot 53° 14' = 9·8734302,

find θ where L cot θ = 9·8734523.

10.   Given L tan 22° 37' = 9·6197205

diff. for $1'$ = 3557,

find the value of

L tan 22° 37' 22"

and the angle whose L tan is 9·6195283.

11. Prove that, θ being any acute angle,

$$L \sin \theta + L \operatorname{cosec} \theta = L \cos \theta + L \sec \theta$$
$$= L \tan \theta + L \cot \theta = 20.$$

12. Given $L \cos 36° 40' = 9\cdot9042411$, find $L \sec 36° 40'$.

13. Given $L \cos 34° 44' = 9\cdot9147729$
$$L \cos 34° 45' = 9\cdot9146852,$$
find the value of $L \cos 34° 44' 27''$.

14. Given $L \sin 36° 40' = 9\cdot7760897$
$$L \cos 36° 40' = 9\cdot9042411,$$
find $L \tan 36° 40'$.

15. Prove that the difference of tabular logarithms of any two ratios is equal to the difference of the logarithms of those two ratios.

16. If $\sin \theta = \cdot8$, find θ,

given $\log 2 = \cdot3010300,$
$$L \sin 53° \ 7' = \ 9\cdot9030136$$
$$L \sec 36° 52' = 10\cdot0968916.$$

17. Find the value of
$$\frac{\sin 34° \ 17' \times \cos 77° \ 23'}{\tan 27° \ 12'}.$$

given $L \sin 12° 37' = 9\cdot3393$
$$L \cos 55° 43' = 9\cdot7507$$
$$L \tan 62° 48' = 10\cdot2891$$
and　　　$\log 23\cdot94 = \ 1\cdot3791$

———

PROPERTIES OF TRIANGLES

81. In a triangle ABC, there are six parts, the three sides and the three angles. It is usual to denote the angles of the triangle by A, B, C and the corresponding opposite sides by a, b, c. The six parts are not independent of one another. The various relations existing among them are deduced in the following articles.

82. *In any triangle, prove that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$
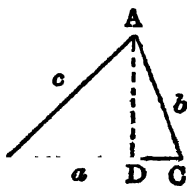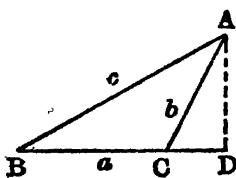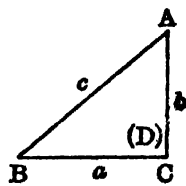
Fig. (i)   Fig. (ii)   Fig. (iii)

Let ABC be any triangle. From A draw AD perpendicular to BC or BC produced if necessary [*Fig. (ii)*]

[In Fig. (i), C is an *acute* angle, in Fig. (ii), C is an *obtuse* angle, in Fig. (iii), C is a *right* angle.]

From $\triangle ABD$, $AD = AB \sin ABD = c \sin B$.

From $\triangle ACD$, $AD = AC \sin ACD = b \sin C$ [*Fig. (i)*]

$$\text{or,} \quad = b \sin (\pi - C) \ [\ Fig. \ (ii) \ ]$$

$$i.e., \quad = b \sin C.$$

$$\therefore \quad b \sin C = c \sin B, \quad i.e., \quad \frac{}{\sin B} \quad \frac{}{\sin C}$$

Similarly, by drawing a perpendicular from B upon CA, we have $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$.

In Fig. (iii), C is a *right* angle ;

$$\sin A = \frac{a}{c} \; ; \; \sin B = \frac{b}{c} \; ; \; \sin C = 1.$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = c = \frac{c}{\sin C}.$$

Hence, in all cases,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \cdots \quad (1)$$

Thus, in any triangle,
the sides are proportional to the sines of the opposite angles.

An alternative method of proof.
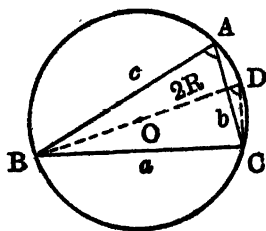
Let O be the centre and R be the radius of the circle circumscribing the triangle ABC.

Join BO and produce it to meet the circumference in D. Join CD. The $\angle BCD$ is then a right angle.

From $\triangle BCD$, $\sin BDC = \dfrac{BC}{BD} = \dfrac{a}{2R}$.

But $\angle BDC = \angle A$, being in the same segment,

$$\therefore \quad \frac{a}{2R} = \sin A, \quad \text{or,} \quad \frac{a}{\sin A} = 2R.$$

Similarly, by joining AO and producing it to meet the circumference in E, and joining CE, BE, it can be shown that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

$$\therefore \quad \mathbf{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.} \qquad \cdots \quad (2)$$

Note 1. If angle A be obtuse, A and D fall on opposite sides of BC and $ABCD$ being cyclic, sin $BDC = \sin (180° - A) = \sin A$, and the same result follows. In case A is a right angle, evidently $2R = a = a/\sin A$, and we get the same result.

Note 2. It follows from the relation (2) that

$$a = 2R \sin A, \, b = 2R \sin B, \, c = 2R \sin C ;$$

$$\sin A = \frac{a}{2R}, \, \sin B = \frac{b}{2R}, \, \sin C = \frac{c}{2R}.$$

83.   *In any triangle, to prove that*

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A,} \text{ or, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B,} \text{ or, } \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C,} \text{ or, } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Take the figures of Art. 82.

First, let C be an *acute* angle [*Fig.* (*i*)] ; then from Geometry,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Now, from $\triangle ACD$, $CD = AC \cos C = b \cos C$.

$$\therefore \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

Next, let the angle C be an *obtuse* angle [*Fig.* (*ii*)] ; then from Geometry,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Now, from $\triangle ACD$, $CD = AC \cos ACD$

$$= b \cos (\pi - C) = - b \cos C.$$

$\therefore$ $c^2 = a^2 + b^2 - 2ab \cos C.$

Lastly, let C be a *right* angle [*Fig. (iii)*] ; then from Geometry,

$$AB^2 = BC^2 + CA^2,$$

i.e., $c^2 = a^2 + b^2 = a^2 + b^2 - 2ab \cos C.$

$$[\because \quad \cos C = \cos 90° = 0.]$$

Hence, for all values of C, we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

$\therefore$ $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}.$

Similarly, the other two relations can be established.

Obs. This theorem expresses the cosines of the angles of a triangle in terms of the sides.

84.   *In any triangle, to prove that*

$$\textbf{a} = \textbf{b} \cos \textbf{C} + \textbf{c} \cos \textbf{B}.$$
$$\textbf{b} = \textbf{c} \cos \textbf{A} + \textbf{a} \cos \textbf{C}.$$
$$\textbf{c} = \textbf{a} \cos \textbf{B} + \textbf{b} \cos \textbf{A}.$$

Take the figures of Art. 82.

In Fig. (i), where C is an *acute* angle,

$$BC = BD + CD$$
$$= AB \cos ABD + AC \cos ACD.$$

$\therefore$ $a = c \cos B + b \cos C.$

In Fig. (ii), where C is an *obtuse* angle,

$$BC = BD - CD$$
$$= AB \cos ABD - AC \cos ACD$$
$$= c \cos B - b \cos (180° - C).$$
$$= c \cos B + b \cos C.$$

In Fig. (iii), where C is a *right* angle,

$$BC = AB \cos B.$$

$\therefore \quad a = c \cos B = c \cos B + b \cos C.$

$$[\because \cos C = \cos 90° = 0.]$$

Thus, in all cases,

$$a = b \cos C + c \cos B.$$

Similarly, the other two relations can be established.

85. From Art. 83 and note of Art. 82, it follows that

$$\tan A = \frac{\sin A}{\cos B} = \frac{\dfrac{a}{2R}}{\dfrac{b^2 + c^2 - a^2}{2bc}} = \frac{abc}{R} \cdot \frac{1}{b^2 + c^2 - a^2}$$

Similarly, $\tan B = \dfrac{abc}{R} \cdot \dfrac{1}{c^2 + a^2 - b^2}.$

$$\tan C = \frac{abc}{R} \cdot \frac{1}{a^2 + b^2 - c^2}.$$

86. Trigonometrical ratios of half angles of a triangle in terms of the sides.

We have, $2 \sin^2 \dfrac{A}{2} = 1 - \cos A = 1 - \dfrac{b^2 + c^2 - a^2}{2bc}.$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}.$$

Let s denote the semi-perimeter of the triangle ;

$$\text{then} \quad 2s = a + b + c.$$

Now, $\quad a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b),$

$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c).$$

Hence, $2 \sin^2 \dfrac{A}{2} = \dfrac{2(s - b) \cdot 2(s - c)}{2bc}$

i.e., $\quad \sin^2 \dfrac{A}{2} = \dfrac{(s - b)(s - c)}{bc}.$

$$\therefore \quad \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

The positive value of the square root must be taken for A, being an angle of a triangle, is less than $180°$; and hence, $\frac{1}{2}A < 90°$ and consequently, $\sin \frac{1}{2}A$ must always be positive.

Again, $2 \cos^2 \dfrac{A}{2} = 1 + \cos A$

$$= 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Now, $b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a)$.

$\therefore \qquad 2 \cos^2 \dfrac{A}{2} = \dfrac{2s \cdot 2(s-a)}{2bc}$, $i.e.$, $\cos^2 \dfrac{A}{2} = \dfrac{s(s-a)}{bc}$.

$\therefore \qquad \cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}.$

Here also the positive value of the square root must be taken ; for $\frac{1}{2}A$ being less than $90°$, $\cos \frac{1}{2}A$ is always positive.

Again, $\tan \dfrac{A}{2} = \sin \dfrac{A}{2} + \cos \dfrac{A}{2}$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}} + \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly, the trigonometrical ratios of $\dfrac{B}{2}$, $\dfrac{C}{2}$ can be obtained in terms of the sides.

Note. *Without assuming the values of $\sin \frac{1}{2}A$, $\cos \frac{1}{2}A$,* the value of $\tan \frac{1}{2}A$ can be obtained by substituting the values of $\cos A$ in terms of the sides from Art. 83 in the relation $\tan^2 \frac{1}{2}A = \dfrac{1 - \cos A}{1 + \cos A}$ and then extracting the square root after simplification.

Thus, we have

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \quad \dots \qquad \dots \ (1)$$

$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \quad \dots \qquad \dots \ (2)$$

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \quad \dots \qquad \dots \ (3)$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

87. Sine of an angle of a triangle in terms of the sides.

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}. \quad [\textit{Art. 86}\,]$$

$$\therefore \quad \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly, $\sin B = \dfrac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}.$

$$\sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}.$$

$\sqrt{s(s-a)(s-b)(s-c)}$, being the expression for the area of the triangle [See *Art. 88*], is usually denoted by the Greek letter $\triangle$. Hence, the above formulæ may be written as

$$\sin A = \frac{2\triangle}{bc}, \ \sin B = \frac{2\triangle}{ca}, \ \sin C = \frac{2\triangle}{ab}.$$

88. Area of a triangle.
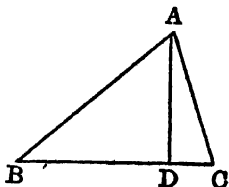
Let *ABC* be a triangle and let $\triangle$ denote its area. Draw *AD* perpendicular to *BC* ; then from $\triangle ACD$,

$$AD = AC \sin C = b \sin C.$$

Now, $\triangle = \frac{1}{2}BC.AD = \frac{1}{2}ab \sin C.$

Similarly by drawing perpendiculars from *B* and *C* to the opposite sides, it can be shown that

$$\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B.$$

Otherwise, $\triangle = \frac{1}{2}ab \sin C$

$\qquad\qquad = \frac{1}{2}ca \sin B \ [\ \because \ b \sin C = c \sin B \ ]$

$\qquad\qquad = \frac{1}{2}bc \sin A \ [\ \because \ a \sin B = b \sin A \ ]$

Thus, $\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C$ $\cdots$ (i)

$\qquad = \frac{1}{2}(product \ of \ two \ sides) \times (sine \ of \ included \ angle).$

Again, $\triangle = \frac{1}{2}bc \sin A = bc \sin \frac{A}{2} \cos \frac{A}{2}$

$\qquad = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$

$\qquad = \sqrt{s(s-a)(s-b)(s-c)}.$ $\cdots$ $\cdots$ (ii)

Substituting in the expression $s = \frac{1}{2}(a+b+c)$, we get

$$\triangle = \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}$$

$$= \frac{1}{4}\{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4\}^{\frac{1}{2}}. \quad \cdots \quad \text{(iii)}$$

Again,

$$\triangle = \frac{1}{2}bc\sin A = \frac{1}{2}bc.\frac{a}{2R} \; [\; Art.\; 82\;] = \frac{\mathbf{abc}}{\mathbf{4R}}. \quad \cdots \quad \text{(iv)}$$

Note. In some text books, S is used to denote the area of a triangle, but to avoid confusion between S and s in writing, the symbol $\triangle$ is preferable.

89. *In any triangle, to prove that*

$$\tan\frac{\mathbf{B-C}}{\mathbf{2}} = \frac{\mathbf{b-c}}{\mathbf{b+c}}\cot\frac{\mathbf{A}}{\mathbf{2}}.$$

We have, in any triangle,

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\therefore \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\dfrac{B+C}{2}\sin\dfrac{B-C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}$$

$$= \cot\frac{B+C}{2}\tan\frac{B-C}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B-C}{2}\left[\; \because \; \frac{A}{2} + \frac{B+C}{2} = 90°\;\right]$$

$$\therefore \quad \tan\frac{B-C}{2} = \frac{b-c}{b+c}.\frac{1}{\tan\dfrac{A}{2}} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

Similarly,

$$\tan\frac{\mathbf{C-A}}{\mathbf{2}} = \frac{\mathbf{c-a}}{\mathbf{c+a}}\cot\frac{\mathbf{B}}{\mathbf{2}}, \; \tan\frac{\mathbf{A-B}}{\mathbf{2}} = \frac{\mathbf{a-b}}{\mathbf{a+b}}\cot\frac{\mathbf{C}}{\mathbf{2}}.$$

90. The three sets of formulæ in Arts. 82, 83, 84 have been established directly from the figures. These three sets

however, are not independent, for, from any one set, the other two sets can be deduced.

For example, let us deduce the formulæ of Art. 83 from those of Art. 84.

By Art. 84, $a = b \cos C + c \cos B$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A.$$

Multiplying these in succession by a, b, and c, and subtracting the first result from the sum of the other two, we have,

$$b^2 + c^2 - a^2 = b(c \cos A + a \cos C) + c(a \cos B + b \cos A)$$
$$- a(b \cos C + c \cos B) = 2bc \cos A.$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}; \text{ similarly, for } \cos B, \cos C.$$

Note. For other cases, see *Appendix*.

91. In working out identities which involve both the sides and angles of a triangle, it is sometimes convenient to express the sides in terms of the angles or the angles in terms of the sides.

Also, it is sometimes found convenient to express the values of $\tan \dfrac{A}{2}$, $\tan \dfrac{B}{2}$, $\tan \dfrac{C}{2}$ in a form in which the denominator is constant and numerator is free from radical. Thus, multiplying the numerator and the denominator of the value of $\tan \dfrac{A}{2}$ by $\sqrt{(s-b)(s-c)}$ and noting that

$$\sqrt{s(s-a)(s-b)(s-c)} = \triangle, \text{ we have}$$

$$\tan \frac{A}{2} = \frac{(s-b)(s-c)}{\triangle}; \text{ similarly, } \tan \frac{B}{2} = \frac{(s-c)(s-a)}{\triangle};$$

$$\tan \frac{C}{2} = \frac{(s-a)(s-b)}{\triangle}.$$

Again, multiplying the numerator and the denominator of the value of $\cot \frac{A}{2}$ by $\sqrt{s(s-a)}$ we have

$$\cot \frac{A}{2} = \frac{s(s-a)}{\triangle}.$$

Similarly, $\cot \frac{B}{2} = \frac{s(s-b)}{\triangle}$; $\cot \frac{C}{2} = \frac{s(s-c)}{\triangle}$.

Ex. 1. *Show that in any triangle,*

$a (\sin B - \sin C) + b (\sin C - \sin A) + c(\sin A - \sin B) = 0.$

Left side $= (a \sin B - b \sin A) + (b \sin C - c \sin B)$
$$+ (c \sin A - a \sin C)$$

$$= 0 + 0 + 0 \left[\because \text{ by Art. 82, } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right]$$

$$= 0.$$

Ex. 2. *Show that in any triangle,*

$$a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$$
$$[H. S. 1961]$$

$a = 2R \sin A$ [*by Art. 82*] $= 2R \sin (B + C),$
$$[\because A + B + C = \pi]$$

$\therefore$ 1st term of the left side $= 2R \sin (B + C) \sin (B - C)$
$$= 2R (\sin^2 B - \sin^2 C).$$
$$[by Ex. 2, Art. 35]$$

Similarly,   2nd term $= 2R (\sin^2 C - \sin^2 A)$
$$3\text{rd term} = 2R (\sin^2 A - \sin^2 B).$$

Now adding together the three terms, the required result follows.

Ex. 3. *In any triangle, prove that*

$$(b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0.$$

Substituting the values of $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$, as given in Art. 91, we have, the left side

$$= (b-c).\frac{s\,(s-a)}{\triangle} + (c-a).\frac{s\,(s-b)}{\triangle} + (a-b).\frac{s\,(s-c)}{\triangle}$$

$$\frac{s}{\triangle}\Big[(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)\Big]$$

$$= \frac{s}{\triangle}\cdot 0 = 0.$$

Ex. 4. *If the cosines of two of the angles of a triangle are inversely proportional to the opposite sides, show that the triangle is either isosceles or right-angled.*

We have, by the question,

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{\sin B}{\sin A}. \qquad [\,by\ Art.\ 82\,]$$

$\therefore$ $\sin A \cos A = \sin B \cos B$, or, $\sin 2A = \sin 2B$,

or, $\sin 2A - \sin 2B = 0$,

or, $2\cos(A+B)\sin(A-B) = 0$.

$\therefore$ either $\cos(A+B) = 0$, *i.e.*, $(A+B) = 90°$,

i.e., the triangle is right-angled ;

or, $\sin(A-B) = 0$, *i.e.*, $A-B = 0$, *i.e.*, $A = B$,

i.e., the triangle is isosceles.

Ex. 5. *If the sides of a triangle are in A.P., show that* $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ *are also in A.P.*

$\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ are in A.P.,

if $\cot\dfrac{B}{2} - \cot\dfrac{A}{2} = \cot\dfrac{C}{2} - \cot\dfrac{B}{2}$.

i.e.,   if $\dfrac{s(s-b)}{\triangle} - \dfrac{s(s-a)}{\triangle} = \dfrac{s(s-c)}{\triangle} - \dfrac{s(s-b)}{\triangle}$,

i.e.,   if $(s-b)-(s-a)=(s-c)-(s-b)$,

i.e.,   if $a-b=b-c$,

i.e.,   if $a,\ b,\ c$ are in A.P.

Ex. 6.   *Show that*

$$b^2 \sin 2C + c^2 \sin 2B = 4\triangle.$$

Left side $= b^2.2 \sin C \cos C + c^2.2 \sin B \cos B$

$\qquad\quad = 2b \sin C.b \cos C + 2c \sin B.c \cos B$

$\qquad\quad = 2b \sin C\,(b \cos C + c \cos B)$

$$[\ \because\ c \sin B = b \sin C\ ]$$

$\qquad\quad = 2ab \sin C \qquad\qquad [\ by\ Art.\ 84\ ]$

$\qquad\quad = 4.\tfrac{1}{2}ab \sin C = 4\triangle. \qquad [\ by\ Art.\ 88\ ]$

Examples XIV(a)

In any triangle, prove that *(Ex. 1 to 21)* :—

1. $\sin \dfrac{B-C}{2} = \dfrac{b-c}{a} \cos \dfrac{A}{2}.$ $\quad$ [*H. S. 1961 Comp.*]

2. $\cos \dfrac{B-C}{2} = \dfrac{b+c}{a} \sin \dfrac{A}{2}.$ $\qquad$ [*H. S. 1961 Comp.*]

3. $(b+c) \cos A + (c+a) \cos B + (a+b) \cos C = a+b+c.$

4. $\dfrac{a+b}{a-b} = \tan \dfrac{A+B}{2} \cot \dfrac{A-B}{2}.$

5. $a^2 + b^2 + c^2 = 2(bc \cos A + ca \cos B + ab \cos C).$

6. $(b+c-a) \tan \dfrac{A}{2} = (c+a-b) \tan \dfrac{B}{2} = (a+b-c) \tan \dfrac{C}{2}.$

7. $\dfrac{a \sin (B-C)}{b^2 - c^2} = \dfrac{b \sin (C-A)}{c^2 - a^2} = \dfrac{c \sin (A-B)}{a^2 - b^2}.$

8. $a^2 (\sin^2 B - \sin^2 C) + b^2 (\sin^2 C - \sin^2 A)$
$$+ c^2 (\sin^2 A - \sin^2 B) = 0.$$

9. $a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A)$
$$+ c^2 (\cos^2 A - \cos^2 B) = 0.$$

10. $\dfrac{a^2 \sin (B-C)}{\sin B + \sin C} + \dfrac{b^2 \sin (C-A)}{\sin C + \sin A} + \dfrac{c^2 \sin (A-B)}{\sin A + \sin B} = 0.$

11. $a \sin \dfrac{A}{2} \sin \dfrac{B-C}{2} + b \sin \dfrac{B}{2} \sin \dfrac{C-A}{2}$
$$+ c \sin \dfrac{C}{2} \sin \dfrac{A-B}{2} = 0.$$

12. $\dfrac{b^2 - c^2}{a^2} \sin 2A + \dfrac{c^2 - a^2}{b^2} \sin 2B + \dfrac{a^2 - b^2}{c^2} \sin 2C = 0.$

13. $a^3 \sin (B-C) + b^3 \sin (C-A) + c^3 \sin (A-B) = 0.$

14. $a^3 \cos (B-C) + b^3 \cos (C-A) + c^3 \cos (A-B) = 3abc.$

15. $\dfrac{a^2 \sin (B-C)}{\sin A} + \dfrac{b^2 \sin (C-A)}{\sin B} + \dfrac{c^2 \sin (A-B)}{\sin C} = 0.$

16. $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$

17. $\dfrac{b^2 - c^2}{\cos B + \cos C} + \dfrac{c^2 - a^2}{\cos C + \cos A} + \dfrac{a^2 - b^2}{\cos A + \cos B} = 0.$

18. $(s-a) \tan \dfrac{A}{2} = (s-b) \tan \dfrac{B}{2} = (s-c) \tan \dfrac{C}{2}.$

19. $\dfrac{b-c}{a} \cos^2 \dfrac{A}{2} + \dfrac{c-a}{b} \cos^2 \dfrac{B}{2} + \dfrac{a-b}{c} \cos^2 \dfrac{C}{2} = 0.$

20. $bc \cos^2 \dfrac{A}{2} + ca \cos^2 \dfrac{B}{2} + ab \cos^2 \dfrac{C}{2} = s^2.$

21. $\dfrac{1}{a} \cos^2 \dfrac{A}{2} + \dfrac{1}{b} \cos^2 \dfrac{B}{2} + \dfrac{1}{c} \cos^2 \dfrac{C}{2} = \dfrac{s^2}{abc}.$

22. If A be $60°$, show that $b + c = 2a \cos \dfrac{B-C}{2}.$

23. Show that a triangle having its sides equal to 3, 5, 7 is an obtuse-angled triangle and determine the obtuse angle.

24. Given $(a + b + c)(b + c - a) = 3bc$, find A.

25. If $c^4 - 2(a^2 + b^2) c^2 + a^4 + a^2 b^2 + b^4 = 0$, prove that
$$C = 60°, \text{ or, } 120°.$$

26. If $a^4 + b^4 + c^4 = 2c^2 (a^2 + b^2)$, prove that
$$C = 45°, \text{ or, } 135°.$$

27. The sides of a triangle are $2x + 3$, $x^2 + 3x + 3$, $x^2 + 2x$; show that the greatest angle is $120°$.

28. If $\dfrac{1}{a + c} + \dfrac{1}{b + c} = \dfrac{3}{a + b + c}$, show that $C = 60°$.

29. If $a = 2b$ and $A = 3B$, find the angles of the triangle.

30. If the cosines of two of the angles of a triangle are proportional to the opposite sides, show that the triangle is isosceles.

31. If $\cos A = \dfrac{\sin B}{2 \sin C}$, show that the triangle is isosceles.

32. If $(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B)$, prove that the triangle is either isosceles or right-angled.

✓**33.** If $(\cos A + 2 \cos C) : (\cos A + 2 \cos B) = \sin B : \sin C$, prove that the triangle is either isosceles or right-angled.

34. If a^2, b^2, c^2 be in A.P., prove that $\cot A$, $\cot B$, $\cot C$ are also in A.P.

35. If $a \cos^2 \dfrac{C}{2} + c \cos^2 \dfrac{A}{2} = \dfrac{3b}{2}$, show that the sides of the triangle are in A.P.

36. If $\sin A : \sin C = \sin (A - B) : \sin (B - C)$, show that a^2, b^2, c^2 are in A.P.

37. If a, b, c are in A.P., show that

$\cos A \cot \frac{1}{2}A$, $\cos B \cot \frac{1}{2}B$, $\cos C \cot \frac{1}{2}C$ are in A.P.

[$\cos A \cot \frac{1}{2}A = (1 - 2 \sin^2 \frac{1}{2}A) \cot \frac{1}{2}A = \cot \frac{1}{2}A - \sin A$.]

38. Assuming $\triangle = \frac{1}{2}bc \sin A$ and using the value of $\cos A$ in terms of sides, show that

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

39. Find the area of the triangle whose sides are

$$\frac{y}{z} + \frac{z}{x}, \quad \frac{z}{x} + \frac{x}{y}, \quad \frac{x}{y} + \frac{y}{z}.$$

40. In a triangle, if $a = 13$, $b = 14$, $c = 15$, find its area.

Prove that in any triangle :

41. $\dfrac{a^2 - b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin (A - B)} = \triangle.$

42. $4\triangle (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2.$

43. $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C.$

44. $a \sin B \sin C + b \sin C \sin A + c \sin A \sin B = \dfrac{3\triangle}{R}.$

45. $(a \sin A + b \sin B + c \sin C)^2$
$$= (a^2 + b^2 + c^2)(\sin^2 A + \sin^2 B + \sin^2 C).$$

46. $\dfrac{\cos B \cos C}{bc} + \dfrac{\cos C \cos A}{ca} + \dfrac{\cos A \cos B}{ab} = \dfrac{1}{4R^2}.$

[*Use* $\Sigma \cot B \cot C = 1$; *ex. 2, Ex. X.*]

47. $\dfrac{b^2 - c^2}{a} \cos A + \dfrac{c^2 - a^2}{b} \cos B + \dfrac{a^2 - b^2}{c} \cos C = 0.$

48. $\dfrac{\cos A}{a} + \dfrac{a}{bc} = \dfrac{\cos B}{b} + \dfrac{b}{ca} = \dfrac{\cos C}{c} + \dfrac{c}{ab}.$

49. $4\triangle = a^2 \cot A + b^2 \cot B + c^2 \cot C.$

50. $\left(\dfrac{a^2}{\sin A} + \dfrac{b^2}{\sin B} + \dfrac{c^2}{\sin C} \right) \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2} = \triangle.$

92. Circum-radius of a triangle.

From Art. 82, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \qquad \cdots \quad (i)$$

Hence, $R = \dfrac{a}{2 \sin A} = \dfrac{abc}{2bc \sin A} = \dfrac{abc}{4\triangle}. \qquad \cdots \quad (ii)$

93. In-radius of a triangle.

Let I be the centre and r the radius of the circle inscribed in the triangle ABC; let D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively.
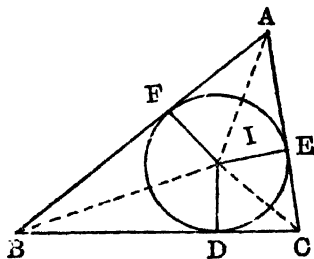
Then, $ID = IE = IF = r$.

Join IA, IB, IC.

$$\begin{aligned}
\triangle ABC &= \triangle IBC + \triangle ICA + \triangle IAB \\
&= \tfrac{1}{2}BC.ID + \tfrac{1}{2}CA.IE + \tfrac{1}{2}AB.IF \\
&= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr \\
&= \tfrac{1}{2}r(a + b + c) = rs.
\end{aligned}$$

Thus, $\triangle = rs$.

$$r = \frac{\triangle}{s}. \qquad \cdots \quad (i)$$

Again, $a = BC = BD + DC$

$$= r \cot \tfrac{1}{2}B + r \cot \tfrac{1}{2}C, \quad \text{from } \triangle^s \ IBD, ICD,$$

$$= r\left[\frac{\cot \tfrac{1}{2}B}{\sin \tfrac{1}{2}B} + \frac{\cos \tfrac{1}{2}C}{\sin \tfrac{1}{2}C}\right]$$

$$= r\left[\frac{\cos \tfrac{1}{2}B \sin \tfrac{1}{2}C + \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}\right]$$

$$= r\frac{\sin (\tfrac{1}{2}B + \tfrac{1}{2}C)}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C} = r\frac{\cos \tfrac{1}{2}A}{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}.$$

$\{ \because \ \tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90°, \sin (\tfrac{1}{2}B + \tfrac{1}{2}C) = \sin (90° - \tfrac{1}{2}A) = \cos \tfrac{1}{2}A. \}$

11

$$= a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \sec \tfrac{1}{2}A = a \, \frac{\sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}{\cos \tfrac{1}{2}A}.$$

Since, by Art. 92(i), $a = 2R \sin A = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$,

∴   **r = 4R sin ½A sin ½B sin ½C.** $\cdots$ (ii)

Since, from the figure, $AF = AE$, $BD = BF$, $CD = CE$ and since the sum of these six quantities is equal to the perimeter,

∴ $AF + BD + CD = $ semi-perimeter $= s$.

i.e., $AF + BC$,   or, $AF + a = s$.

∴ $AF = s - a = AE$.

Similarly, $BF = s - b = BD$; $CE = s - c = CD$.

From $\triangle AIF$, $IF = AF \tan IAF$.

∴   **r = (s − a) tan ½A.** ⎫

Similarly, **r = (s − b) tan ½B.** ⎬ $\cdots$ (iii)

  and **r = (s − c) tan ½C.** ⎭

Note. *Distances of the in-centre from the vertices.*

From $\triangle AIF$, $IA = IF$ cosec IAF. ∴ *IA = r cosec ½A.*

Similarly,   *IB = r cosec ½B* and   *IC = r cosec ½C.*

94. Ex-radii of a triangle.
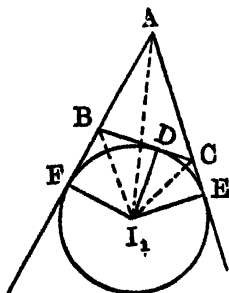
Let I_1 be the centre and r_1 the radius of the escribed circle (opposite to the angle A) of the $\triangle ABC$; let D, E, F be the points of contact of the circle with the sides BC, and AC and AB produced.

Let r_2, r_3 denote the radii of the escribed circles opposite to the angles B and C respectively.

Now, $\quad I_1D = I_1E = I_1F = r_1$; join AI_1, BI_1, CI_1.

$$\triangle ABC = \triangle I_1AB + \triangle I_1AC - \triangle I_1BC$$
$$= \tfrac{1}{2}AB.I_1F + \tfrac{1}{2}AC.I_1E - \tfrac{1}{2}BC.I_1D$$
$$= \tfrac{1}{2}cr_1 + \tfrac{1}{2}br_1 - \tfrac{1}{2}ar_1$$
$$= \tfrac{1}{2}r_1\,(b+c-a) = \tfrac{1}{2}r_1\,(b+c+a-2a)$$
$$= \tfrac{1}{2}r_1\,(2s-2a)$$
$$= r_1\,(s-a).$$

Thus, $\quad \triangle = r_1\,(s-a).$

$\therefore\quad \mathbf{r_1 = \dfrac{\triangle}{s-a}}$

Similarly, $\quad \mathbf{r_2 = \dfrac{\triangle}{s-b}} \qquad$ (i)

and $\quad \mathbf{r_3 = \dfrac{\triangle}{s-c}}$.

Again, $\quad a = BC = BD + CD$

$$= r_1 \cot I_1BD + r_1 \cot I_1CD,$$
$$\text{from } \triangle^s\ I_1BD,\ I_1CD$$
$$= r_1 \cot (90° - \tfrac{1}{2}B) + r_1 \cot (90° - \tfrac{1}{2}C),$$

because, $\angle I_1BD = \tfrac{1}{2}(180° - B) = 90° - \tfrac{1}{2}B,$

and $\quad \angle I_1CD = \tfrac{1}{2}(180° - C) = 90° - \tfrac{1}{2}C.$

$\therefore\quad a = r_1\,(\tan \tfrac{1}{2}B + \tan \tfrac{1}{2}C)$

$$= r_1\left[\dfrac{\sin \tfrac{1}{2}B}{\cos \tfrac{1}{2}B} + \dfrac{\sin \tfrac{1}{2}C}{\cos \tfrac{1}{2}C}\right]$$

$$= r_1\left[\dfrac{\sin \tfrac{1}{2}B \cos \tfrac{1}{2}C + \sin \tfrac{1}{2}C \cos \tfrac{1}{2}B}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}\right]$$

$$= r_1\,\dfrac{\sin (\tfrac{1}{2}B + \tfrac{1}{2}C)}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}$$

$$= r_1\,\dfrac{\cos \tfrac{1}{2}A}{\cos \tfrac{1}{2}B \cos \tfrac{1}{2}C}, \text{ as in Art. 93.}$$

$$r_1 = a \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \sec \tfrac{1}{2}A.$$

Putting $a = 2R \sin A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A$,

$\quad\quad r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

Similarly, $r_2 = 4R \cos \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C$,   (ii)

and $r_3 = 4R \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C$.

Again, $AE = AC + CE = b + CD$ [$\because$ $CE = CD$]

and $AF = AB + BF = c + BD$. [$\because$ $BF = BD$]

But $AE = AF$; therefore, by addition, we get

$\quad 2AE = b + c + BD + CD = b + c + a = 2s$.

$$\therefore \quad AE = s.$$

Again, from $\triangle AI_1E$, $I_1E = AE \tan I_1AE$.

$\quad\quad \therefore \quad r_1 = s \tan \frac{1}{2}A$.

Similarly, $r_2 = s \tan \frac{1}{2}B$,   (iii)

and $r_3 = s \tan \frac{1}{2}C$. }

Note. *Distances of ex-centres from the vertices.*

From $\triangle AI_1F$, $I_1A = I_1F \csc I_1AF$.

$\therefore \quad\quad\quad I_1A = r_1 \csc \frac{1}{2}A$

$\quad\quad\quad\quad\quad = 4R \cos \frac{1}{2}B \cos \frac{1}{2}C$.   [*by formula (ii)*]

From $\triangle BI_1F$, $I_1B = I_1F \csc I_1BF$.

$\therefore \quad\quad\quad I_1B = r_1 \sec \frac{1}{2}B$ [$\because$ $\angle I_1BF = 90° - \frac{1}{2}B$]

Similarly, $I_1C = r_1 \sec \frac{1}{2}C$.

In the same way, $I_2B = r_2 \csc \frac{1}{2}B$, $I_3C = r_3 \csc \frac{1}{2}C$.

Ex. 1. *Prove that* $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$.

By formula (i), Art. 94,

$$\text{left side} = \frac{s-a}{\triangle} + \frac{s-b}{\triangle} + \frac{s-c}{\triangle}$$

$$= \frac{3s-(a+b+c)}{\triangle} = \frac{3s-2s}{\triangle} = \frac{s}{\triangle} = \frac{1}{r}.$$

Ex. 2.   *Prove that* $4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = \dfrac{s}{R}.$

Left side $= 4 . \sqrt{\dfrac{s(s-a)}{bc}} . \sqrt{\dfrac{s(s-b)}{ca}} . \sqrt{\dfrac{s(s-c)}{ab}}$

$\qquad = \dfrac{4s}{abc} . \sqrt{s(s-a)(s-b)(s-c)}$

$\qquad = \dfrac{4s}{abc} \triangle = s . \dfrac{4\triangle}{abc} = \dfrac{s}{R}$ by formula (ii), Art. 92.

Ex. 3.   *Show that*

$$\frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3}.$$

$$r_2 r_3 = \frac{\triangle^2}{(s-b)(s-c)} = s(s-a),$$

$\therefore \quad bc - r_2 r_3 = \frac{1}{4} [4bc - 2s \ (2s - 2a)]$

$\qquad\qquad = \frac{1}{4} [4bc - (a+b+c)(b+c-a)]$

$\qquad\qquad = \frac{1}{4} [4bc + a^2 - (b+c)^2] = \frac{1}{4} [a^2 - (b-c)^2]$

$\qquad\qquad = \frac{1}{4} [(a+b-c)(a-b+c)] = (s-b)(s-c).$

$\therefore \quad \dfrac{bc - r_2 r_3}{r_1} = \dfrac{(s-b)(s-c)}{r_1} = \dfrac{(s-a)(s-b)(s-c)}{\triangle}$

$\qquad\qquad = \dfrac{\triangle}{s} = r.$

Similarly, the other ratios are equal to the same quantity.

Ex. 4.   *Prove that in any triangle,*

$$r_1 + r_2 + r_3 - r = 4R.$$

Left side $= \left(\dfrac{\triangle}{s-a} + \dfrac{\triangle}{s-b} \right) + \left(\dfrac{\triangle}{s-c} - \dfrac{\triangle}{s} \right)$

$\qquad = \triangle . \dfrac{2s - (a+b)}{(s-a)(s-b)} + \triangle . \dfrac{c}{s(s-c)}$

$\qquad = \triangle c \left[\dfrac{1}{(s-a)(s-b)} + \dfrac{1}{s(s-c)} \right] \quad [\because \ 2s = a+b+c. \ ]$

$\qquad = \triangle c \left[\dfrac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right]$

Now, Numerator $= 2s^2 - s(a+b+c) + ab$

$$= 2s^2 - s.2s + ab = ab.$$

Denominator $= \triangle^2$.

$\therefore$ left side $= \dfrac{abc}{\triangle} = 4R$.

Ex. 5. *If* $r_1 = r_2 + r_3 + r$, *prove that the triangle is right-angled.*

From the given relation, we have

$$r_1 - r = r_2 + r_3,$$

or, $\dfrac{\triangle}{s-a} - \dfrac{\triangle}{s} = \dfrac{\triangle}{s-b} + \dfrac{\triangle}{s-c}$,

or, $\dfrac{\triangle.a}{s(s-a)} = \dfrac{\triangle(2s-b-c)}{(s-b)(s-c)} = \dfrac{\triangle.a}{(s-b)(s-c)}$.

$\therefore$ $s(s-a) = (s-b)(s-c)$.

$\therefore$ $\tan^2 \tfrac{1}{2}A = \dfrac{(s-b)(s-c)}{s(s-a)} = 1.$ $\therefore$ $\tan \tfrac{1}{2}A = 1.$

$\therefore$ $\tfrac{1}{2}A = 45°$. $\therefore$ $A = 90°$.

Note. Although we get $\tan \tfrac{1}{2}A = \pm 1$, we reject the negative value because $\tfrac{1}{2}A$ is an acute angle.

Examples XlV(b)

Prove that in any triangle (*Ex. 1 to 14*) :—

1. $\sin A + \sin B + \sin C = \dfrac{s}{R}$.

2. $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$.

[*Use* $\cos A + \cos B + \cos C = 1 + 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$]

3. $\dfrac{b-c}{r_1} + \dfrac{c-a}{r_2} + \dfrac{a-b}{r_3} = 0$.

4. $r_2 r_3 + r_3 r_1 + r_1 r_2 = s^2.$

5. $r = R (\cos A + \cos B + \cos C - 1).$

6. $r_1 = R (\cos B + \cos C - \cos A + 1).$

[*Use* $\cos B + \cos C - \cos A = -1 + 4 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$]

7. $a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \dfrac{\triangle}{R}.$

8. $a \cot A + b \cot B + c \cot C = 2 (R + r).$

$\left[a \cot A = \dfrac{a}{\sin A}. \cos A = 2R \cos A. \quad Then \ use \ Ex. \ 2. \right]$

9. $R = \dfrac{1}{4} \dfrac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)}{r_2 r_3 + r_3 r_1 + r_1 r_2}.$

10. $\triangle = \sqrt{r r_1 r_2 r_3} = r^2 \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C.$

11. $\left(\dfrac{1}{r} - \dfrac{1}{r_1} \right)\left(\dfrac{1}{r} - \dfrac{1}{r_2} \right)\left(\dfrac{1}{r} - \dfrac{1}{r_3} \right) = \dfrac{4R}{r^2 s^2} = \dfrac{16R}{r^2 (a + b + c)^2}.$

[*A. I. 1938*]

12. $\left(\dfrac{1}{r} + \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} \right)^2 = \dfrac{4}{r} \left(\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} \right).$

13. $r_1 (r_2 + r_3) \operatorname{cosec} A = r_2 (r_3 + r_1) \operatorname{cosec} B$

$\qquad\qquad = r_3 (r_1 + r_2) \operatorname{cosec} C.$

14. $\dfrac{bc}{r_1} + \dfrac{ca}{r_2} + \dfrac{ab}{r_3} = 2R \left\{ \dfrac{b}{a} + \dfrac{c}{a} + \dfrac{c}{b} + \dfrac{a}{b} + \dfrac{a}{c} + \dfrac{b}{c} - 3 \right\}.$

15.   In a triangle, $a = 13, \ b = 14, \ c = 15$; find r and R.

16.   If $a, \ b, \ c$ are in A.P., show that $r_1, \ r_2, \ r_3$ are in H.P.

17.   If in a triangle, $3R = 4r$, show that

$\qquad 4 (\cos A + \cos B + \cos C) = 7.$

18.   If the diameter of an ex-circle be equal to the perimeter of the triangle, show that the triangle is right-angled.

[*Use* $r_1 = s \tan \tfrac{1}{2}A.$]

19. If $\left(1 - \dfrac{r_1}{r_2}\right)\left(1 - \dfrac{r_1}{r_3}\right) = 2$, show that the triangle must be right-angled.

20. If $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right-angled.

21. If S be the area of the in-circle and S_1, S_2, S_3 are the areas of the escribed circles, then

$$\frac{1}{\sqrt{S}} = \frac{1}{\sqrt{S_1}} + \frac{1}{\sqrt{S_2}} + \frac{1}{\sqrt{S_3}}.$$

22. In any triangle, prove that the area of the in-circle is to the area of the triangle as $\pi : \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C$.

23. If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle to the opposite sides, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

24. If x, y, z be the lengths of the perpendiculars from the circum-centre on the sides BC, CA, AB of the triangle ABC, prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}.$$

25. If x, y, z are respectively equal to IA, IB, IC, and α, β, γ are respectively equal to I_1A, I_2B, I_3C, show that

(i) $\dfrac{xyz}{abc} = \dfrac{r}{s}$.   (ii) $\dfrac{x}{\alpha} + \dfrac{y}{\beta} + \dfrac{z}{\gamma} = 1$.

(iii) $\dfrac{bc}{\alpha^2} + \dfrac{ca}{\beta^2} + \dfrac{ab}{\gamma^2} = 1$.   (iv) $ax^2 + by^2 + cz^2 = abc$.

[*Use Notes of Arts. 93 and 94.*]

26. Prove that
$$(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$$

27. If $\dfrac{r}{2} = \dfrac{r_1}{12} = \dfrac{R}{5}$ prove that the $\triangle$ is right angled.

28. If I be the in-centre and x, y, z be the circumradii of the triangles BIC, CIA, AIB, then show that
$$xyz = 2R^2 r.$$

29. If I be the in-centre of the triangle ABC, show that
$$IA.IB.IC = \frac{abc}{s} \triangle$$

where $\triangle$ denotes the area of $\triangle ABC$ and s is its semi-perimeter.

30. If D, E, F be the points of contact of the in-circle with the sides BC, CA, AB respectively, show that
$$EF = 2(s - a) \sin \tfrac{1}{2}A,$$
$$FD = 2(s - b) \sin \tfrac{1}{2}B,$$
$$DE = 2(s - c) \sin \tfrac{1}{2}C.$$

31. If the lengths of the internal bisectors of the angle of a triangle be equal, show that the triangle is isosceles.

32. If x, y, z be the lengths of the internal bisetors of the angles of a triangle and l, m, n be the lengths of those bisectors, produced to meet the circle, show that

(i) $\dfrac{\cos \tfrac{1}{2}A}{x} + \dfrac{\cos \tfrac{1}{2}B}{y} + \dfrac{\cos \tfrac{1}{2}C}{z} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

(ii) $l \cos \tfrac{1}{2}A + m \cos \tfrac{1}{2}B + n \cos \tfrac{1}{2}C = a + b + c.$

33. If the internal bisectors of the angles of a triangle make angles a, β and γ with the sides BC, CA and AB respectively, show that
$$a \sin 2a + b \sin 2\beta + c \sin 2\gamma = 0.$$

34. If l, m, n be the lengths of the medians of a triangle ABC, prove that

(i) $(b^2 - c^2)l^2 + (c^2 - a^2)m^2 + (a^2 - b^2)n^2 = 0.$

(ii) $4(l^2 + m^2 + n^2) = 3(a^2 + b^2 + c^2).$

35. Perpendiculars AL, BM, CN are drawn from the vertices A, B, C of an acute-angled triangle on the opposite sides and produced to meet the circumcircle of $\triangle ABC$ in L', M', N'. If LL', MM', NN' be equal to α, β, γ respectively, show that

$$\text{(i)} \quad \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C).$$

$$\text{(ii)} \quad \frac{AL'}{AL} + \frac{BM'}{BM} + \frac{CN'}{CN} = 4.$$

36. If x, y, z are respectively equal to IA, IB, IC and x_1, y_1, z_1 are respectively equal to I_1A, I_2B, I_3C where I is the in-centre and I_1, I_2, I_3 are then ex-centres of $\triangle ABC$, then show that

$$\text{(i)} \quad \frac{b-c}{ax_1^2} + \frac{c-a}{by_1^2} + \frac{a-b}{cz_1^2} = 0.$$

$$\text{(ii)} \quad x^2\left(\frac{1}{b} - \frac{1}{c}\right) + y^2\left(\frac{1}{c} - \frac{1}{a}\right) + z^2\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

37. Prove that in the triangle ABC

$$\frac{IA}{I_1A} + \frac{IB}{I_2B} + \frac{IC}{I_3C} = 1.$$

38. Prove that in the triangle ABC

$$a.BP.CP + b.CP.AP + c.AP.BP = abc$$

where P is the orthocentre of the triangle

39. If AL, BM, CN are the perpendiculars from the angular points of $\triangle ABC$ to the opposite sides, show that

$$\frac{1}{AL} + \frac{1}{BM} + \frac{1}{CN} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

40. In the triangle ABC (with the usual notation), show that $\dfrac{r_1 r_2 r_3}{r^3} = \dfrac{(a+b+c)^3}{(b+c-a)(c+a-b)(a+b-c)}.$

SOLUTION OF TRIANGLES

95. In a triangle, there are six parts, the three sides and the three angles. These are not independent, but are connected by the relations between the sides and angles of the triangle, which have been established in Chapter XIV. In fact, if three of the parts are given, the remaining three can, in general, be determined, and the corresponding triangle completely known. The cases that can arise are the following :

(1) three sides may be given ;

(2) three angles may be given ;

(3) two sides and the included angle may be given ;

(4) two angles and one side may be given ;

(5) two sides and an opposite angle may be given.

We shall discuss these cases one by one.

96. Three sides given.

Let the three sides a, b, c of a triangle ABC be given. Now, provided the sum of any two of these given sides is greater than the third, the triangle ABC with the three given sides can be geometrically constructed and the triangle is unique ; in other words, its angles are definite. To determine any angle, say A, we may use the rigorous formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and thereby determine cos A, and then from the cosine-table find out the angle with this cosine. It is clear that the angle, being an angle of a triangle, lies between 0 and π, and within this range an angle with a given cosine has got only one value. Thus the angle is definitely known.

Here we want to make one point clear. Though the formula used is rigorous, the cosine-table, by means of which we determine the angle with a given cosine, gives only approximate values. Now, it is a principle proved in books on higher mathematics (with the aid of Calculus), that *when an angle is determined by using an approximate table the best result is obtained by using the Logarithmic tangent-table*, and an angle determined from its L tan, using a four-figure table is more accurate than that determined by using even a seven-figure sine-table or cosine-table. If a suitable tangent formula is available, therefore, we should make use of it.

Hence, for practical purposes, in this case, to determine A, we use the formula,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}},$$

where $s = \tfrac{1}{2}(a + b + c)$, which is known.

Taking logarithm, and adding 10, we get the value of L tan $\tfrac{1}{2}A$ and therefore A is known.

Similarly, B and C are determined.

In case tan $\tfrac{1}{2}A$ happens to be equal to the tangent of a standard angle, $\tfrac{1}{2}A$ is at once known and the use of logarithm is not wanted.

Ex. *The sides of a triangle are* 2, 3, 4. *Find the greatest angle, having given*

$$log\ 2 = \cdot 30103, \qquad log\ 3 = \cdot 4771213,$$

$$L\ tan\ 52°\ 14' = 10\cdot 1108395, \quad L\ tan\ 52°\ 15' = 10\cdot 1111004.$$

Here, $\qquad s = \dfrac{2+3+4}{2} = \dfrac{9}{2}.$

The greatest side 4 being denoted by 'a', the greatest angle A (which is opposite to the greatest side) is given by

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(\tfrac{9}{2}-2)(\tfrac{9}{2}-3)}{\tfrac{9}{2}(\tfrac{9}{2}-4)}} = \sqrt{\frac{5.3}{9.1}} = \sqrt{\frac{10}{2.3}}.$$

$$\therefore \quad L\ tan\ \tfrac{1}{2}A = 10 + \tfrac{1}{2}\,(\log 10 - \log 2 - \log 3)$$

$$= 10 + \tfrac{1}{2}\,(1 - \cdot 30103 - \cdot 4771213)$$

$$= 10\cdot 1109244.$$

Now, $L\ tan\ \tfrac{1}{2}A$ lies between $L\ tan\ 52°\ 14'$ and $L\ tan\ 52°\ 15'$.

Hence, $\tfrac{1}{2}A$ lies between $52°\ 14'$ and $52°\ 15'$.

Let $\qquad \tfrac{1}{2}A = 52°\ 14'\ x''$.

Then diff. for x'' is $\cdot 0000849$,

and diff. for $1'$ *i.e.*, $60''$ is $\cdot 0002609$.

Hence, $\dfrac{x}{60} = \dfrac{849}{2609}$, or, $x = \dfrac{60 \times 849}{2609} = 19\cdot 5$ nearly.

Hence, $\tfrac{1}{2}A = 52°\ 14'\ 19''\cdot 5$,

or, $\quad A = 104°\ 28'\ 39''$ nearly.

97. Three angles given.

In this case the triangle cannot be solved, for there are innumerable triangles with the same three angles. All these

triangles, being equiangular, are similar, and the ratio of
their sides can be determined from the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

or, $a : b : c = \sin A : \sin B : \sin C.$

Ex. *The angles of a triangle are in the ratio* $2 : 3 : 7$.
Prove that the sides are in the ratio $\sqrt{2} : 2 : (\sqrt{3}+1)$.

The angles being in the ratio $2 : 3 : 7$, and their sum
being $180°$, the angles are evidently $30°$, $45°$ and $105°$
respectively.   Hence, the ratio of the sides will be

$$\sin 30° : \sin 45° : \sin 105°,$$

i.e., $\dfrac{1}{2} : \dfrac{1}{\sqrt{2}} : \dfrac{\sqrt{3}+1}{2\sqrt{2}},$

or, $\sqrt{2} : 2 : (\sqrt{3}+1).$

Examples XV(a)

1.   The sides of a triangle are 24, 22, 14 ; find the least
angle, given $L \tan 17° 33' = 9{\cdot}500042$, diff. for $1' = 439$.

2.   The sides of a triangle are 50, 36 and 28 ; find the
greatest angle, having given

$$\log 19 = 1{\cdot}2787536, \qquad \log 29 = 1{\cdot}4623980$$

$L \tan 51° 0' = 10{\cdot}0916308, \ L \tan 50° 1' = 10{\cdot}0918891.$

3.   The sides of a triangle are 9, 10 and 11 ; find the
angle opposite to the side 10, given

$$L \tan 29° 30' = 9{\cdot}7526420, \ L \tan 29° 29' = 9{\cdot}7523472,$$
$$\log 2 = {\cdot}30103. \qquad\qquad [\ C.\ U.\ 1943\ ]$$

4.   The sides of a triangle are 2, 3, 4.   Find all the
angles correct to degrees and minutes by the help of
mathematical tables.

5. (i) The sides of a triangle are 15, 19, 24 ; find the greatest angle of the triangle.

Given $\log 5\cdot7 = \cdot75587$, $L \cos 88° 59' = 8\cdot24903$
diff. for $1' = 718$.   [*C. U. 1936*]

(ii) Find the greatest angle in degrees, minutes and seconds in a triangle whose sides are 5, 6, 7, having given
$$\log 6 = \cdot7781513$$

$L \cos 39° 14' = 9\cdot8890644$, diff. for $60'' = \cdot0001032$.

6. (i) The sides of a triangle are 7, 8, 9 ; solve the triangle.   [*C. U. 1938*]

(ii) If $a = 35$, $b = 40$, $c = 66$, determine the greatest angle.   [*C. U. 1945*]

[*Use Mathematical Tables*]

7.   Given $a = \sqrt{6}$, $b = 2$, $c = \sqrt{3} - 1$; solve the triangle.

8.   Given $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} + 1$; solve the triangle.

9.   If $a = 7$, $b = 5$, $c = 8$, solve the triangle.
Given $\cos 38° 11' = \frac{11}{14}$.

10.   If $a = 3 + \sqrt{3}$, $b = 2\sqrt{3}$, $c = \sqrt{3}$, solve the triangle.

11.   The angles of a triangle are 105°, 60° and 15° ; find the ratio of the sides.

12.   If $A = 45°$, $B = 60°$, show that $c : a = \sqrt{3} + 1 : 2$.

13.   The angles of a triangle are as $1 : 2 : 7$; find the ratio of the greatest side to the least side.

14.   If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$, find $a : b : c$.

15.   If the angles adjacent to the base of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, show that the altitude is half the base.

16.   If the sides of a triangle are 4, 5, 6, show that the greatest angle is double the least.

98. Two sides and the included angle given.

Let the two sides b, c and the included angle A of a triangle ABC be given. It is easy to construct the triangle geometrically, and there will be only one definite triangle with the given parts. To find the other angles B and C, we notice that

$$B + C = 180° - A,$$

i.e., $$\frac{B+C}{2} = 90° - \frac{A}{2}.$$

Again,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}.$$

$$\therefore \ L \tan \frac{B-C}{C} = 10 + \log \left(\frac{b-c}{b+a} \cot \frac{A}{2} \right)$$

$$= \log \left(\frac{b-c}{b+c} \right) + L \cot \frac{A}{2}.$$

b, c, and A being given, the right-hand side is known and thus, $L \tan \frac{B-C}{2}$ is known, whence $\frac{B-C}{2}$ is known.

Now $\frac{B+C}{2}$ and $\frac{B-C}{2}$ being both known, by addition and subtraction, we get B and C respectively.

The reason of using tangent formula to determine $\frac{B-C}{2}$ is already explained in Art. 96.

When B and C are known, the third side a is easily obtained from

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \ \text{or}, \ = \frac{c}{\sin C}.$$

Ex. *In a triangle, $b = 2·25$, $c = 1·75$, $A = 54°$, find B and C, having given,*

$$\log 2 = ·301030, \qquad\qquad L \tan 63° = 10·292834$$

$$L \tan 13° \ 47' = 9·389724, \qquad L \tan 13° \ 48' = 9·390270.$$

$$[\ C. \ U. \ 1931 \ ]$$

Here,

$$\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 27° = 63°. \qquad \text{(i)}$$

Again,

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{\cdot 5}{4} \cot 27$$

$$= \tfrac{1}{8} \tan 63°.$$

$$\therefore \quad L \tan \frac{B-C}{2} = L \tan 63° - 3 \log 2$$

$$= 10\cdot 292834 - \cdot 903090$$

$$= 9\cdot 389744.$$

Now, $L \tan 13° 47' = 9\cdot 389724$

and $\quad L \tan 13° 48' = 9\cdot 390270.$

Hence, $\dfrac{B-C}{2}$ . . . .° ...' "

we get, diff. for $x'' = \cdot 000020$

and diff. for $1'$ *i.e.*, $60'' = \cdot 000546$,

$$\therefore \quad \frac{x}{60} = \frac{\cdot 20}{546}, \quad \text{or,} \quad x = \frac{20 \times 60}{546} = 2\cdot 2 \text{ nearly.}$$

Hence, $\dfrac{B-C}{2} = 13° 47' 2'''2$ nearly.

Combining with (i), $B = 76° 47' 2'''2$ and $C = 49° 12' 57'''8.$

99. Two angles and a side given.

Let any side a of a triangle ABC, and any two of its angles be given. The sum of the three angles being $180°$, the third angle is also known. To find the other two sides b and c, we use the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Ex. *In a triangle ABC, A = 38° 20′, B = 45° and b = 64 ft. Find c, having given log 2 = ·30103, L sin 83° 20′ = 9·99705 and log ·089896 = $\overline{2}$·95374.*

Here,

$$C = 180° - (A + B) = 180° - 83° \ 20'.$$

Now,

$$\frac{c}{\sin C} = \frac{b}{\sin B},$$

or, $$\frac{c}{\sin (180° - 83° \ 20')} = \frac{64}{\sin 45°} = \frac{64}{1/\sqrt{2}} = 64 \sqrt{2}.$$

$$\therefore \quad c = 2^{\frac{13}{2}} \sin 83° \ 20'.$$

$$\therefore \quad \log c = \tfrac{13}{2} \log 2 + L \sin 83° \ 20' - 10$$

$$= \tfrac{13}{2} \ ('30103) + 9'99705 - 10 = 1'95374.$$

Thus, log c has the same mantissa as log ˙089896, but has its characteristic 1.   Hence, $c = 89˙896$ feet.

Examples XV(b)

1.   Two sides of a triangle are 3 and 5 feet and the included angle is 120° ; find the other angles, having given

$$\log 4˙8 = ˙6812412$$

L tan 8° 12' = 9˙1586706, diff. for 60″ ÷ 8940.

2.   If $b = 1300$, $c = 1400$ and $A = 60°$, find B and C.
Given log 3 = ˙4771213,
L tan 3° 40' = 8˙8067422, diff. for 10″ = 3306.

3.   If $a = 21$, $b = 11$, $C = 34° \ 42' \ 30''$, find A and B.
Given log 2 = ˙30103,
and L tan 72° 38' 45″ = 10˙50515.

4.   If the sides a and b are in the ratio 7 : 3 and the included angle C is 60°, find A and B, given

log 2 = ˙3010300,   log 3 = ˙4771213
L tan 34° 42' = 9˙8403776, diff. for 1' = 2699.

5. Two sides of a plane triangle are 14 and 11 and the included angle is 60°. Find the remaining angles, having given $L \tan 11° 44' = 9\cdot3174299$, $L \tan 11° 45' = 9\cdot3180640$.

[*C. U. 1922*]

6. (i) Two sides of a triangle are 80 and 100 ft. and the included angle is 60°. Find the other angles. [*C. U. 1946*]

(ii) If $a = 5$, $b = 3$, $C = 70° 30'$, find the remaining angles.

(iii) If $a = 39\cdot9$, $b = 43\cdot2$, $C = 38° 14'$, solve the triangle.

[*Use Mathematical Tables*]

7. (i) In a plane triangle, $b = 540$, $c = 420$ and $A = 52° 6'$; find B and C, having given

$L \tan 26° 3' \ = 9\cdot6891430$,

$L \tan 14° 20' = 9\cdot4074189$,

$L \tan 14° 21' = 9\cdot4079453$.   [*C. U. 1934*]

(ii) Given $a = 70$, $b = 35$, $C = 36° 52' 12''$, $\log 3 = 0\cdot4771213$, $L \cot 18° 26' 6'' = 10\cdot4771213$. Calculate the other two angles A and B.   [*C. U. 1935, '37*]

8. If $a = 2\sqrt{6}$, $c = 6 - 2\sqrt{3}$, $B = 75°$, solve the triangle.

9. Two sides of a triangle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ and the included angle is 60°; solve the triangle.

10. (i) If $a = 2$, $b = 1 + \sqrt{3}$, $C = 60°$, solve the triangle.

(ii) If $a = 2$, $b = 4$, $C = 60°$, find A and B.

11. If $a = 19$, $B = 52° 28'$ and $C = 93° 40'$, find b, having given $\log 27038 = 4\cdot4319746$; $\log 19 = 1\cdot2787536$;

$\log 27037 = 4\cdot4319585$;

$L \sin 52° 28' = 9\cdot8992727$, $L \sin 33° 52' = 9\cdot7460595$.

12. If $B = 45°$, $C = 10°$ and $a = 200$ ft., find b, having given
$\log 2 = \cdot30103$, $L \sin 55° = 9\cdot9133645$,
$\log 1726\cdot4 = 3\cdot2371414$, $\log 1726\cdot5 = 3\cdot2371666$.

[*C. U. 1947*]

13. If $A = 41° 13' 22''$, $B = 71° 19' 5''$, and $a = 55$, find b,
given $\log 55 = 1\cdot7403627$, $\log 79063 = 4\cdot8979775$,

$\qquad L \sin 41° 13' 22'' = 9\cdot8188779$,

$\qquad L \sin 71° 19' \ 5'' = 9\cdot9764927$.

14. (i) If $B = 70° 30'$, $C = 78° 10'$, $a = 102$, solve the
triangle.

(ii) If $a = 39$, $A = 81° 35'$, $B = 27° 55'$, solve the triangle.

(iii) If $A = 37° 15'$, $B = 72° 5'$, $a = 75\cdot2$, find b and c.

[*Mathematical tables should be used*]

15. If $A = 75°$, $B = 30°$, $b = \sqrt{8}$, solve the triangle.

16. If $A = 30°$, $B = 45°$, $b = 2$, solve the triangle.

17. In a triangle in which each base angle is double of
the third angle, the base is 2 ; solve the triangle.

18. Given $a = \sqrt{57}$, $A = 60°$, $\triangle = 2\sqrt{3}$, find b and c.

100. Two sides and an opposite angle given.

Let the two sides b and c in a triangle ABC, and the
angle B opposite to the side b be given.

In this case, we get the angle C from the formula,

$$\frac{\sin C}{c} = \frac{\sin B}{b}, \text{ or, } \sin C = \frac{c \sin B}{b}.$$

Now, three cases may arise, namely.

(i) $c \sin B > b$. In this case $\sin C$ is greater than 1,
and so C cannot be obtained. In fact in this
case no triangle is possible.

(ii) $c \sin B = b$. Here, $\sin C$ becomes 1 and therefore,
$C = 90°$. Thus, $A = 90° - B$. We thus get a right-
angled triangle with right angle at C, and the
side a will be obtained from

$$c^2 = a^2 + b^2, \text{ or, } a = \sqrt{c^2 - b^2}.$$

(iii) $c \sin B < b$. In this case $\sin C$ is less than 1, and hence, C can be determined. Now, sines of supplementary angles are known to be equal, and an angle of a triangle may be acute or obtuse. We therefore get two supplementary values of C having the same value for $\sin C$. Three sub-cases now arise :

Sub-case A. If of the two given sides, $b > c$, then $B > C$, and therefore the obtuse value of C becomes inadmissible, for otherwise B is also obtuse and two angles B and C of the triangle become both obtuse. Thus, the only admissible solution is the acute value of C. Now, B and C being both known, A is obtained from $A + B + C = 180°$. The side a will be known from

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

Thus, the triangle is uniquely solved.

Sub-case B. If $b = c$, then $B = C$, and here also the obtuse value of C is inadmissible ; with the acute value of C the triangle is uniquely solved exactly as in the above case.

Sub-case C. If $b < c$, then $B < C$, so that C may be either acute or obtuse. Both the supplementary values of C being admissible now, there will be two possible triangles with the three given parts b, c, B. Corresponding to each value of C, the value of A is determined from $A + B + C = 180°$, and then a is obtained from the formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or, } \frac{c}{\sin C}.$$

As there are two solutions of the triangle in this case, each equally admissible, this sub-case in the solution of a triangle in which b, c, B are given and $b > c \sin B$ and $< c$, is referred to as the **Ambiguous Case** in the solution of triangles.

We may sum up the results as follows :

When in a triangle, b, c, B are given,

    (i) if $b < c \sin B$, no triangle is possible;

    (ii) if $b = c \sin B$, we get a definite right-angled triangle as solution ;

    (iii) if $b > c$ and therefore necessarily $> c \sin B$, we get one definite solution having C acute ;

    (iv) if $b = c$ and therefore necessarily $> c \sin B$, we get one definite solution having C acute ;

    (v) if $b > c \sin B$ but $< c$, there are two solutions, and this case is the *ambiguous case*.

101.   Geometrical treatment of the Ambiguous Case.

To make the ideas clear, we proceed to construct geo-
metrically the triangle in which two sides and an opposite
angle, *viz.*, b, c and B are given.
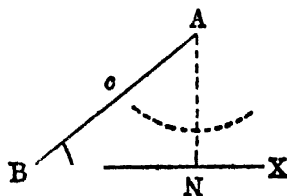
Fig. (i)   Fig. (ii)
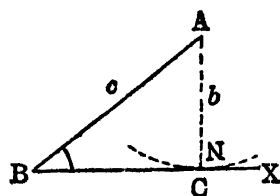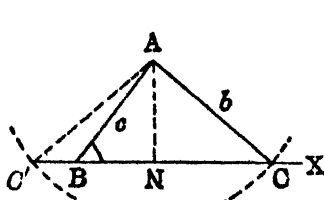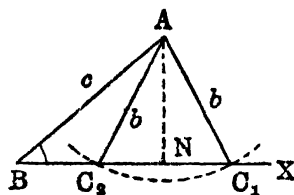
Fig. (iii)   Fig. (iv)

Let ABX be the given angle B, and along one arm of it, take $AB = c$. Let AN be the perpendicular from A on BX. Then $\dfrac{AN}{AB} = \sin B$, so that $AN = AB \sin B = c \sin B$.

With centre A and radius b draw a circle.

Case (i). If $b < c \sin B$, *i.e.*, $< AN$, the circle does not meet the side BX at all and no triangle is therefore obtained. [See *fig.* (i)]

Case (ii). If $b = c \sin B$, *i.e.*, $= AN$, the circle touches the side BX at C coincident with N, as in fig. (ii). Hence, a right-angled triangle is formed, in which the sides AB, AC and the angle B have the given values c, b, B. Thus, ABC is the required triangle.

Case (iii). If $b > c$, *i.e.*, $> AB$, the circle cuts BX at two points C and C' on opposite sides of B as in fig. (iii). The triangle ABC', though it has the sides AB, AC equal to the given quantities c and b, has the angle B not equal to the given angle, but equal to its supplement. Hence it is not the solution required. In this case the triangle ABC is the only solution.

Case (iv). If $b = c$, *i.e.*, $= AB$, the point C' of the above case coincides with B, and only one triangle ABC is obtained as the required solution.

Case (v). If $b > c \sin B$, *i.e.*, $> AN$ but less than c (or, AB), the circle cuts BX at two points C_1 and C_2 on the same side of B as in fig. (iv). Both the triangles ABC_1 and ABC_2 have the same three given parts and both are possible solutions. This is therefore the *Ambiguous case*.

Note. By considering the equation

$$b^2 = c^2 + a^2 - 2ac \cos B$$

in which b, c, B are given, we may first of all determine a, instead of trying to determine C.

Considering the equation as a quadratic in a, *viz.*,

$$a^2 - 2c \cos B . a + c^2 - b^2 = 0,$$

and by solving it, we tet

$$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B}.$$

(i) If $b < c \sin B$, $b^2 - c^2 \sin^2 B$ is negative and thus the two values of a are imaginary. (*No solution*)

(ii) If $b = c \sin B$, $b^2 - c^2 \sin^2 B = 0$ and thus the two values of a are real and coincident.

(*one solution* : one triangle right-angled at C, since $b = c \sin B$)

(iii) If $b > c \sin B$, $b^2 - c^2 \sin^2 B$ is positive, so two values of a are real and distinct, but they are not always admissible.

(a) When $b > c$, {*i.e.*, $b^2 > c^2 (\sin^2 B + \cos^2 B)$}, $b^2 - c^2 \sin^2 B > c^2 \cos^2 B$, *i.e.*, $\sqrt{b^2 - c^2 \sin^2 B} > c \cos B$ and hence one value of a is positive and the other negative. (*one solution*)

(b) When $b = c$, $b^2 - c^2 \sin^2 B = c^2 - c^2 \sin^2 B = c^2 \cos^2 B$ and hence one value of a is zero. (*one solution*)

(c) When $b < c$, *i.e.*, $b^2 < c^2 (\sin^2 B + \cos^2 B)$, $b^2 - c^2 \sin^2 B < c^2 \cos^2 B$, *i.e.*, $\sqrt{b^2 - c^2 \sin^2 B} < c \cos B$.

So both values of a are real and positive. (*two solutions*)

This is known as the *algebraical discussion* of the ambiguous case.

An example illustrating the algebraic method is added below.

Ex. 1. *In a triangle*, $b = 15$ ft., $c = 10$ ft., $B = 60°$. *Find a and A having given* $\sin 84° \; 44' = \cdot 99578$.

We have $b^2 = c^2 + a^2 - 2ca \cos B$, giving here

$$225 = 100 + a^2 - 20a \cos 60° ;$$

or, $a^2 - 10a - 125 = 0$ whence

$$a = 5 \pm 5 \sqrt{6}.$$

Rejecting the negative value for a as inadmissible, the only possible value of $a = 5 (\sqrt{6} + 1)$ ft. There is thus one solution and there is no ambiguity. In fact this is case (iii) of the previous article.

$$\text{Again,}\quad \sin A = \frac{a}{b} \sin B = \frac{5(\sqrt{6} + 1)}{15} \cdot \frac{\sqrt{3}}{2} = \frac{3 \sqrt{2} + \sqrt{3}}{6}$$

$$= \frac{3 \times 1\cdot41421 \cdots + 1\cdot73205}{6} = \cdot99578 \cdots$$

so $\qquad A = 84^\circ\ 44'.$

Ex. 2. *In a triangle, $a = 73\cdot4$, $b = 64\cdot9$ and $B = 48^\circ\ 13'\ 25''$; find A, having given*

$$log\ 734 = 2\cdot8656961, \qquad log\ 649 = 2\cdot8122447$$
$$L\ \sin 48^\circ\ 13'\ 25'' = 9\cdot8725936$$
$$L\ \sin 57^\circ\ 30' = 9\cdot9260292 \qquad (\ diff.\ for\ 1' = 804\ )$$

Is the case ambiguous ?

Here,

$$\sin A = \frac{a \sin B}{b} = \frac{734}{649} \sin 48^\circ\ 13'\ 25''.$$

$\therefore \qquad L \sin A = \log 734 - \log 649 + L \sin 48^\circ\ 13'\ 25''$
$$= 2\cdot8656961 - 2\cdot8122447 + 9\cdot8725936$$
$$= 9\cdot9260450.$$

Now, diff. of this from $L \sin 57^\circ\ 30' = 158$ (*i.e.*, $\cdot0000158$) and diff. for $1'$ (or $60''$) $= 804$ (*i.e.*, $\cdot0000804$).

Hence, $\quad A = 57^\circ\ 30'\ x''$ where

$$\frac{x}{60} = \frac{158}{804} \text{ whence } x = 11\cdot8 \text{ nearly.}$$

Thus, $A = 57^\circ\ 30'\ 11\cdot8''$ or its supplement $122^\circ\ 29'\ 48\cdot2''$ which has also the same sine, and so the same L sine.

Now, in this case $a > b$ and so $A > B$ and thus both values of A are admissible. The case is, therefore, the *ambiguous case* and will have two solutions.

Examples XV(c)

1. Given (i) $A = 30°$, $a = 6$, $b = 4$.

        (ii) $A = 60°$, $a = 7$, $b = 8$.

       (iii) $A = 45°$, $a = 2$, $b = 8$.

      (iv) $A = 30°$, $a = 3$, $b = 6$.

Find in which case the solution is ambiguous, in which case there is one solution, and in which case there is no solution.

2. (i) If $b = 2$, $c = \sqrt{3} + 1$ and $B = 45°$, solve the triangle.

  (ii) If $a = 3$, $b = 3\sqrt{3}$, $A = 30°$, find B.

3. If $a = 2$, $b = \sqrt{6}$, $B = 60°$, solve the triangle.

4. If $a = 2$, $b = 5$, $A = 30°$, solve the triangle.

5. If b, c, B are given and if $b < c$, show that
$$(a_1 - a_2)^2 + (a_1 + a_2)^2 \tan^2 B = 4b^2,$$

a_1 and a_2 being the two possible values of a.

6. In the ambiguous case, given a, b and A, prove that the difference between the two values of c is
$$2\sqrt{a^2 - b^2 \sin^2 A}.$$

7. If a, b, A are given, and if c_1, c_2 are the values of the third side in the ambiguous case, prove that if $c_1 > c_2$,

  (i) $c_1 - c_2 = 2a \cos B_1$.   [B. H. U. I. 1928]

  (ii) $c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A = 4a^2 \cos^2 A$.

                   [B. H. U. I. 1935 ; Pat. I. 1936]

  (iii) $\cos \dfrac{C_1 - C_2}{2} = \dfrac{b \sin A}{a}$.   [A. I. 1941]

8. If $b = 16$, $c = 25$ and $B = 33° 15'$, find the other angles ; given

    $L \sin 33° 15' = 9\cdot7390129$, $\log 2 = \cdot30103$,

    $L \sin 58° 57' = 9\cdot9328376$, $L \sin 58° 56' = 9\cdot9327616$.

9. If $a = 5$, $b = 4$, $A = 45°$, find B and C ; given
$$\log 2 = \cdot30103, \; L \sin 34° \; 27' = 9\cdot75257.$$

10. If $a = 30$, $b = 300$, find A in order that B may be a right angle, having given that
$$L \sin 5° \; 44' = 8\cdot9995595, \text{ diff. for } 1' = 12565.$$

11. If $a = 16$, $c = 25$ and $C = 60°$, find the other angles ; given
$$\log 2 = \cdot30103, \qquad \log 3 = \cdot4771213$$
$$L \sin 33° \; 39' = 9\cdot7436024, \text{ diff. for } 1' = 1897.$$

12. If $b = 165$, $c = 258$, and $B = 35° \; 10'$, find the angles A and C ; given
$$\log 1\cdot65 = \cdot21749, \qquad \log 2\cdot58 = \cdot41162$$
$$L \sin 35° \; 10' = 9\cdot76039, \quad L \sin 64° \; 14' = 9\cdot95452.$$

13. If $2b = 3a$ and $\tan^2 A = \frac{3}{5}$, prove that there are two values of the third side, one of which is double the other.

14. If A_1, B_1 and A_2, B_2 are the angles of the two triangles in the ambiguous case where b, c, C are given,

then $\dfrac{\sin A_1}{\sin B_1} + \dfrac{\sin A_2}{\sin B_2} = 2 \cos C$.

15. Show that in the case that admits of two solutions, the two values of C satisfy the equation
$$\frac{(a + b)^2}{1 + \cos C} + \frac{(b - a)^2}{1 - \cos C} = \frac{2a^2}{\sin^2 A} \cdot \quad [\; B. \; H. \; U. \; I. \; 1942 \;]$$

16. If $\log b + 10 = \log c + L \sin B$, can the triangle be ambiguous ?

Miscellaneous Examples II

In any triangle ABC, prove that (*Ex. 1 to 8*) :—

1. $\dfrac{1}{a}\cos A + \dfrac{1}{b}\cos B + \dfrac{1}{c}\cos C = \dfrac{a^2 + b^2 + c^2}{2abc}$.

2. $(b^2 + c^2 - a^2)\tan A = (c^2 + a^2 - b^2)\tan B$
$$= (a^2 + b^2 - c^2)\tan C.$$

3. $b^2 + c^2 - 2bc\cos(A + 60°) = c^2 + a^2 - 2ca\cos(B + 60°)$
$$= a^2 + b^2 - 2ab\cos(C + 60°).$$

4. $\left(\cot\tfrac{1}{2}A - \tan\tfrac{1}{2}B - \tan\tfrac{1}{2}C\right)^{\frac{1}{2}}$

$+ \left(\cot\tfrac{1}{2}B - \tan\tfrac{1}{2}C - \tan\tfrac{1}{2}A\right)^{\frac{1}{2}} + \left(\cot\tfrac{1}{2}C - \tan\tfrac{1}{2}A - \tan\tfrac{1}{2}B\right)^{\frac{1}{2}}$

$$= \left(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\right)^{\frac{1}{2}}.$$

5. $a\sin(B - C)\cos(B + C - A) + b\sin(C - A)$
$$\times\cos(C + A - B) + c\sin(A - B)\cos(A + B - C) = 0.$$

6. $\dfrac{a\sin A + b\sin B + c\sin C}{a\cos A + b\cos B + c\cos C} = \dfrac{R}{abc}(a^2 + b^2 + c^2)$.

7. $(b + c - 2a)\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(B - C)$
$$+ (c + a - 2b)\sin\tfrac{1}{2}B\sin\tfrac{1}{2}(C - A)$$
$$+ (a + b - 2c)\sin\tfrac{1}{2}C\sin\tfrac{1}{2}(A - B) = 0.$$

8. $a\cos A\cos 2A + b\cos B\cos 2B + c\cos C\cos 2C$
$$+ 4\cos A\cos B\cos C\,(a\cos A + b\cos B + c\cos C) = 0.$$

9. If in a triangle, a^2, b^2, c^2 are in A P., show that $\tan A$, $\tan B$, $\tan C$ are in H.P.

10. If in a triangle, $\sin A$, $\sin B$, $\sin C$ are in H.P., show that $1 - \cos A$, $1 - \cos B$, $1 - \cos C$ are in H.P.

11. Determine the triangle whose sides are three consecutive terms in the series of natural numbers and whose largest angle is double the least.

12. If in a triangle, $\cos 3A + \cos 3B + \cos 3C = 1$, show that one angle must be $120°$.

13. If in a triangle, $\sin A, \sin B, \sin C$ be in A.P., show that $\tan \frac{1}{2}A \tan \frac{1}{2}C = \frac{1}{3}$.

14. If $a = 5$, $b = 7$ and $A = 30°$, find B in degrees and minutes, having given

$$\sin 44° = 0\cdot6947, \ \sin 45° = 0\cdot7071.$$

15. In the *ambiguous case*, the area of one of the triangles is n times that of the other ; show that if b be the greater of the given sides and c the less, $\dfrac{b}{c}$ is less than $\dfrac{n+1}{n-1}$.

16. In the *ambiguous case*, show that the circum-circles of the two triangles are equal.

17. Prove that

(i) $\tan^{-1}\left(\dfrac{x \cos \phi}{1 - x \sin \phi}\right) - \tan^{-1}\left(\dfrac{x - \sin \phi}{\cos \phi}\right) = \phi.$

(ii) $\tan^{-1}\dfrac{t_1 - t_2}{1 + t_1 t_2} + \tan^{-1}\dfrac{t_2 - t_3}{1 + t_2 t_3}$

$\qquad + \cdots + \tan^{-1}\dfrac{t_{n-1} - t_n}{1 + t_{n-1}t_n} = \tan^{-1}t_1 - \tan^{-1}t_n.$

18. If the sum of four angles be $180°$, prove that the sum of the products of their cosines taken two and two together is equal to the sum of the products of their sines taken similarly.

19. Prove that $\cos^2 A + \cos^2\left(A + \dfrac{\pi}{3}\right) + \cos^2\left(A - \dfrac{\pi}{3}\right) = \dfrac{3}{2}$.

[*C. U. 1933*]

20. In a triangle ABC, if $\tan \dfrac{A}{2}$, $\tan \dfrac{B}{2}$, $\tan \dfrac{C}{2}$ be in Arithmetical Progression then $\cos A, \cos B, \cos C$ are also in Arithmetical Progression.

21. Give in general terms the solutions of the following equation :

$$\tan (x+b) \tan (x+c) + \tan (x+c) \tan (x+a)$$
$$+ \tan (x+a) \tan (x+b) = 1.$$

22. If $A+B+C = 180°$, prove that

$$\left(1 + \tan \frac{A}{4} \right)\left(1 + \tan \frac{B}{4} \right)\left(1 + \tan \frac{C}{4} \right)$$
$$= 2 \left(1 + \tan \frac{A}{4} \tan \frac{B}{4} \tan \frac{C}{4} \right).$$

23. Prove that

$$\sin^2 x + \sin^2 y + \sin^2 z + \sin^2 (x+y+z)$$
$$= 2 - 2 \cos (x+y) \cos (y+z) \cos (z+x).$$

24. Solve the following equation :

$$\tan x + \tan \left(x + \frac{\pi}{3} \right) + \tan \left(x + \frac{2\pi}{3} \right) = 3.$$

[*Left side reduces to* 3 tan 3x.]

25. Prove that in a triangle ABC,

$$\triangle = \frac{(a+b+c)^2}{4 \left(\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} \right)}.$$

26. Prove that

$$\log \sin 8x = 3 \log 2 + \log \sin x + \log \cos x$$
$$+ \log \cos 2x + \log \cos 4x.$$

27. Show that in any triangle ABC,

$$\log \tan \frac{A}{2} = \tfrac{1}{2} \ [\log (s-b) + \log (s-c) - \log s - \log (s-a)].$$

28. Prove that (i) $x^{\log y} = y^{\log x}$.

(ii) $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1.$

29. In any right-angled triangle ABC, C being the right angle, show that $R + r = \tfrac{1}{2} (a+b)$.

30. Show how to solve a triangle having given the three perpendiculars from the vertices on the opposite sides.

GRAPHS OF TRIGONOMETRICAL FUNCTIONS

102. Changes in the Trigonometrical ratios of an angle *as the angle increases from* $0°$ *to* $360°$.
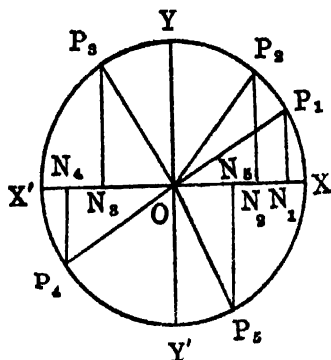
Suppose an angle traced out by a revolving line starting from OX, changes gradually from $0°$ to $360°$.

Take a circle with centre O of any radius. It is clear that in determining the trigonometrical ratios of an angle XOP_1 in its different positions, we can keep the hypotenuse OP_1 always the same, equal to the radius of the circle.

(i) Changes in sine.

When the angle N_1OP_1 ($= \theta$ say) is zero, its sine is zero. As the angle increases from $0°$ to $90°$, the hypotenuse OP_1 remaining the same, the opposite side P_1N_1 is positive and gradually increases, as is evident by comparing the triangles N_1OP_1 and N_2OP_2.

Hence, $\sin \theta = \dfrac{P_1 N_1}{OP_1}$ gradually increases, until when $\theta = 90°$, $P_2 N_2$ and, OP_2 both coincide with OY and $\sin \theta$ attains its greatest value 1.

As θ still further increases, from 90° to 180°, the hypotemuse OP_3 retains the same value, but $P_3 N_3$ remaining positive, now gradually diminishes from OY to zero, and so $\sin \theta$ diminishes from 1 to 0. In the third quadrant, as θ increases from 180° to 270°, $P_4 N_4$ is negative and numerically increases from zero to OY', the hypotenuse remaining always positive and unaltered. Sin θ is therefore negative and numerically increases from 0 to 1 ; in other words, it diminishes gradually from 0 to -1. In the fourth quadrant, as θ increases from 270° to 360°, $P_5 N_5$ remaining negative numerically diminishes from OY' to 0, and $\sin \theta$ therefore remaining negative numerically diminishes from 1 to 0 ; in other words, it increases from -1 to 0. The results are therefore as follows :

In the *first* quadrant, as θ increases from 0° to 90°,

$\sin \theta$ *increases from* 0 *to* 1.

In the *second* quadrant, as θ increases from 90° to 180°,

$\sin \theta$ *diminishes from* 1 *to* 0.

In the *third* quadrant, as θ increases from 180° to 270°,

$\sin \theta$ *diminishes from* 0 *to* -1.

In the *fourth* quadrant, as θ increases from 270° to 360°,

$\sin \theta$ *increases from* -1 *to* 0.

(ii) Changes in cosine.

In the first quadrant, as the angle XOP_1 increases, ON_1 diminishes, from the value of OX at $\theta = 0°$ to the value 0 at $\theta = 90°$, and is always positive.

In the second quadrant, as θ goes on increasing from 90° to 180°, ON_3 increases numerically from 0 to OX' but is

negative. In the third quadrant, ON_4 remains negative, but diminishes numerically from OX' to 0. In the fourth quadrant, ON_5 is positive and increases from 0 to OX again.

The hypotenuse remains always positive and is equal to OX or OX' in magnitude.

We thus come to the conclusions :

As θ increases *from* 0° *to* 90°,

$\cos \theta$ *diminishes from* 1 *to* 0.

As θ increases *from* 90° *to* 180°,

$\cos \theta$ *diminishes from* 0 *to* -1.

As θ increases *from* 180° *to* 270°,

$\cos \theta$ *increases from* -1 *to* 0.

As θ increases *from* 270° *to* 360°,

$\cos \theta$ *increases from* 0 *to* 1.

(iii) Changes in tangent.

As θ goes on increasing from 0° to 90° in the first quadrant, $P_1 N_1$ increases from 0 to OY and simultaneously ON_1 decreases from OX to 0, both remaining positive : hence,

$\tan \theta = \dfrac{P_1 N_1}{ON_1}$ increases from the value $\dfrac{O}{OX} = 0$ to $\dfrac{OY}{O} \to \infty$.

In the second quadrant, $P_3 N_3$ diminishes from OY to 0 while ON_3, becoming negative, numerically increases from 0 to OX'. Hence, $\tan \theta = \dfrac{P_3 N_3}{ON_3}$ is negative but numerically diminishes from ∞ to 0, *i.e.*, increases from $-\infty$ to 0.

Immediately before 90°, $\tan \theta$ is positive and very large, while immediately after 90° $\tan \theta$ is negative and numerically very large. In fact, here, as θ passes through the value 90° from the first to the second quadrant, there is

a sudden break or discontinuity in the value of tan θ, which suddenly passes from a very large positive value to a very large negative value, *i e.*, from the positive to the negative side in passing through infinity.

In the third quadrant, both P_4N_4 and ON_4 are negative and P_4N_4 increases numerically from 0 to OY' while ON_4 decreases numerically from OX' to 0. Hence, $\tan \theta = \dfrac{P_4N_4}{ON_4}$ is positive and increases from 0 to ∞.

In the fourth quadrant, P_5N_5 is negative and numerically diminishes from OY' to 0 while ON_5 is positive and increases from 0 to OX. Hence, $\tan \theta = \dfrac{P_5N_5}{ON_5}$ is negative and numerically diminishes from ∞ to 0, *i.e.*, increases from $-\infty$ to 0.

In passing through 270°, there is another discontinuity, tan θ suddenly passing from the positive to the negative side through infinity.

The results are therefore as follows :

As θ increases *from* 0° *to* 90°, tan θ *increases from* 0 *to* ∞

As θ passes through 90°, tan θ suddenly changes from
$+ \infty$ to $- \infty$

As θ increases *from* 90° *to* 180°, tan θ *increases from*
$- \infty$ *to* 0

As θ increases *from* 180° *to* 270°, tan θ *increases from*
0 *to* ∞

As θ passes through 270°, tan θ suddenly changes from
$+ \infty$ to $- \infty$

As θ increases *from* 270° *to* 360°, tan θ *increases from*
$- \infty$ *to* 0.

(iv) Changes in cotangent.

From the changes in the value of the tangent the changes in cot $\theta = \frac{1}{\tan \theta}$ are traced as follows :

θ increasing *from* 0° *to* 90°, cot θ *diminishes from*
$$\infty \ to \ 0$$

θ increasing *from* 90° *to* 180°, cot θ *diminishes from*
$$0 \ to - \infty$$

As θ passes through 180°, there is a sudden change in
$$\text{cot } \theta \text{ from } - \infty \text{ to } + \infty$$

θ increasing *from* 180° *to* 270°, cot θ *diminishes from*
$$+ \infty \ to \ 0$$

θ increasing *from* 270° *to* 360°, cot θ *diminishes from*
$$0 \ to - \infty$$

As θ passes through 360°, cot θ again suddenly changes
$$\text{from} - \infty \text{ to } + \infty.$$

(v) Changes in secant.

For sec $\theta = \frac{1}{\cos \theta}$, the results are as follows :

From 0° *to* 90° for θ, sec θ *increases from* 1 *to* ∞.

Here, there is a sudden change from $+ \infty$ to $- \infty$.

Then *from* 90° *to* 180°, sec θ *increases from* $- \infty$ *to* $- 1$.

From 180° *to* 270°, sec θ *diminishes from* $- 1$ *to* $- \infty$.

Here, again there is a sudden change from $- \infty$ to $+ \infty$.

Then *from* 270° *to* 360°, sec θ *diminishes from* ∞ *to* 1.

(vi) Changes in cosecant.

For cosec $\theta = \frac{1}{\sin \theta}$, the results are as follows :

From 0° *to* 90°, cosec θ *diminishes from* ∞ *to* 1.

From 90° *to* 180°, cosec θ *increases from* 1 *to* ∞.

Here, cosec θ suddenly changes from $+ \infty$ to $- \infty$.

13

Then *from* 180° *to* 270°, cosec θ *increases from*

$$-\infty \ to \ -1.$$

From 270° *to* 360°, cosec θ *diminishes from* − 1 *to* − ∞.

As θ passes through 360°, cosec θ again suddenly

changes from − ∞ to + ∞.

Note. As θ increases by complete multiples of 2π (*i.e.*, 360°) we know that all the trigonometrical ratios remain unaltered. Hence after 360°, as θ goes on increasing, the same series of values for the ratios are repeated over and over again for each complete revolution of the revolving line. The trigonometrical ratios are therefore all of them **periodic functions** having the **same period** 2π,* after each of which the same cycle of values is repeated.

The changes traced out above, of the trigonometrical ratios, may be much more clearly demonstrated to the eye from a study of their graphs.

103.   Graphs of Trigonometrical Functions.

Just like algebraic functions, trigonometrical functions (*i.e.*, sin x, cos x, $\sin^2 2x + \tan \dfrac{x}{2}$, etc.) may be conveniently represented by means of graphs, showing their changes with the change in the values of the angles.

The method is the same as for graphs in Algebra.   Two straight lines XOX' and YOY', intersecting at right angles are taken as axes of co-ordinates.   Along the x-axis, the angles are represented on a suitably chosen scale, positive angles along OX, and negative angles along OX'.   Along the y-axis the values of the trigonometrical functions corresponding to the angles are represented on a suitably chosen scale, positive values being measured upwards (along OY), and negative values downwards (along OY').   Thus, the *abscissa* and *ordinate* of a point stand respectively for an angle and its trigonometrical function.

* tan θ and cot θ have a period π.

Plotting a number of points in this way and joining them free-hand, we get the required graph of a given trigonometrical function.

104. Graph of sin x *or sine-graph.*

Let $y = \sin x$.

Using the table of natural sines, the corresponding values of x and y are tabulated corresponding to the values of x differing by $10°$ (the values of y being correct up to two places of decimals) as follows :—

x	$-90°$	$-80°$	$-70°$	$-60°$	$-50°$	$-40°$	$-30°$	$-20°$	$-10°$	$0°$
y or $\sin x$	-1	$-·98$	$-·94$	$-·87$	$-·77$	$-·64$	$-·50$	$-·34$	$-·17$	0

x	$10°$	$20°$	$30°$	$40°$	$50°$	$60°$	$70°$	$80°$	$90°$	$100°$	$110°$	$120°$	etc.
y or $\sin x$	$·17$	$·34$	$·50$	$·64$	$·77$	$·87$	$·94$	$·98$	1	$·98$	$·94$	$·87$	etc.

Now, let the scale be so chosen that 1 small division along OX represents $10°$, and 10 small divisions along OY represent unity.[*]

The points corresponding to the tabulated values are plotted on the graph paper according to the scale chosen and joined free-hand.

The graph is as shown on the next page (drawn here between the range $x = -180°$ to $x = +360°$).

*According to the graph paper supplied and the range within which the graph is to be drawn, the scale should be suitably chosen in each individual case separately.
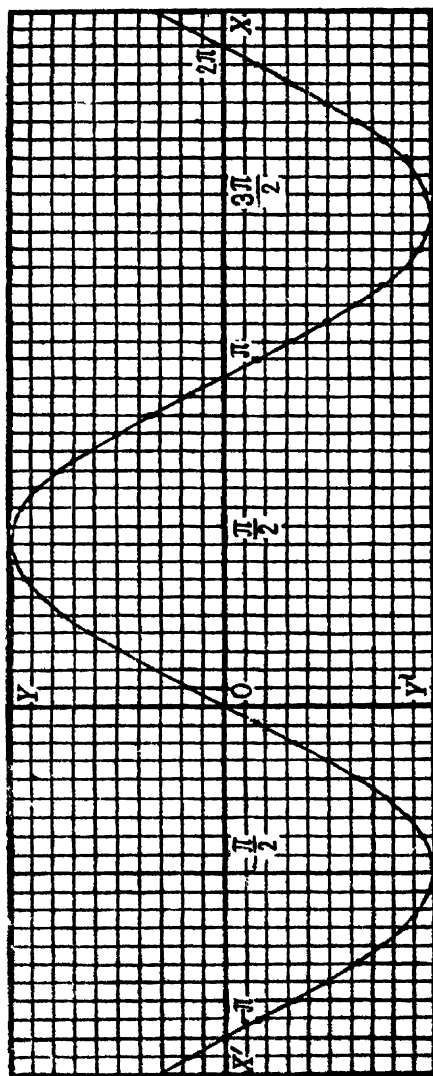
Sine graph

Note 1. In the table of natural sines, sines of angles from 0° to 90° only are available. With the help of the formulæ sin $(-\theta) = -\sin\theta$, sin $(180° - \theta) = \sin\theta$, sin $(180° + \theta) = -\sin\theta$ etc. of Chapter IV, however, the tabulation for sin θ shown above, outside the range of 0° to 90° is effected.

Similar is the case of tabulation for other graphs in the following pages.

Note 2. *Peculiarities of the sine-graph.*

From the figure, the following features will be apparent :—(i) the graph is continuous, and wavy in form ; (ii) the maximum value of sin x is $+1$ and the minimum value is -1, these values being attained for values of x which are odd multiples of 90° ; (iii) sin x is 0 at the origin and at points for which x is an even multiple of 90° *i.e.*, any multiple of 180° ; (iv) that sin $\left(\dfrac{\pi}{2} - x\right) = \sin\left(\dfrac{\pi}{2} + x\right)$, sin $(\pi - x) = \sin x$, sin $(-x) = -\sin x$, sin $(\pi + x) = -\sin x$ etc. ; (v) since sin $(2r\pi + x) = \sin x$, the portion between 0 to 2π is repeated over and over again on either side.

105. Graph of cos x or *cosine-graph.*

Let $y = \cos x$.

Using the table of natural cosines (see *Note 1 of the previous Article*), the corresponding values of x and y are tabulated at intervals of 10° for x as follows :—

x	$-90°$	$-80°$	$-70°$	$-60°$	$-50°$	$-40°$	$-30°$	$-20°$	$-10°$
y or cos x	0	·17	·34	·50	·64	·77	·87	·94	·98

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	110°	etc.
y or cos x	1	·98	·94	·87	·77	·64	·50	·34	·17	0	$-$·17	$-$·34	etc.
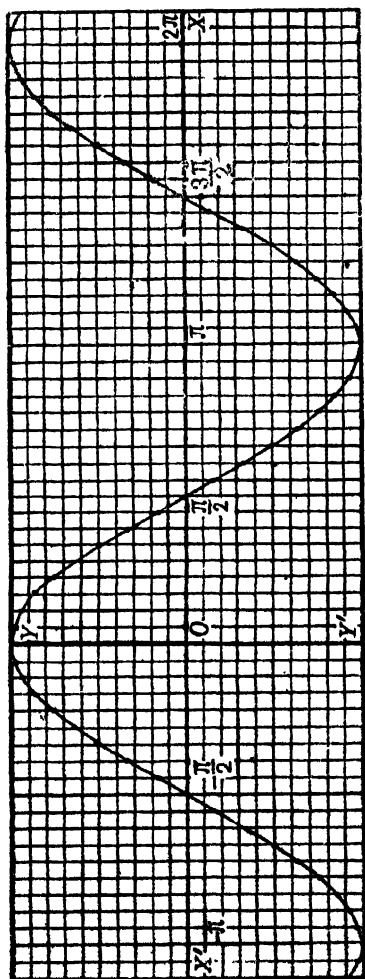
Cosine-graph

Now, choosing the scale such that 1 small division along
OX represents 10°, and 10 small divisions along OY repre-
sent unity, the points corresponding to the above tabulated
values are plotted and joined free-hand.

We then get the required graph, which is shown on the
annexed page (shown here between the range $-\pi$ to $+2\pi$
of x).

Note. It is apparent from the figure, that the *cosine-graph is
exactly the same as the sine-graph only shifted wholesale backwards
(to the left) through a space of* 90°.

This is due to the fact that sin $(90° + x) = \cos x$, or sin $x = \cos (x - 90°)$
so that the ordinate in the sine-graph corresponding to any value of
$x =$ the ordinate of the cosine-graph corresponding to a value of x which
is 90° less than before.

106. Graph of tan x or *tangent-graph*.

Let $y = \tan x$.

Using the table of natural tangents, the corresponding
values of x and y are tabulated at intervals of 10° of x
as follows :—

x	$-20°$	$-10°$	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	etc.
y or tan x	$-\cdot36$	$-\cdot18$	0	$\cdot18$	$\cdot36$	$\cdot58$	$\cdot84$	1·19	1·73	2·75	5·67	∞	$-5\cdot67$	etc.

Now, choosing the scale such that 1 small division along
OX represents 10°, and 3 small divisions along OY represent
unity, the points corresponding to the above tabulated values
are plotted and joined free-hand.

The graph is as shown on the next page (shown here
between the range $-\pi$ to $+2\pi$ for x).
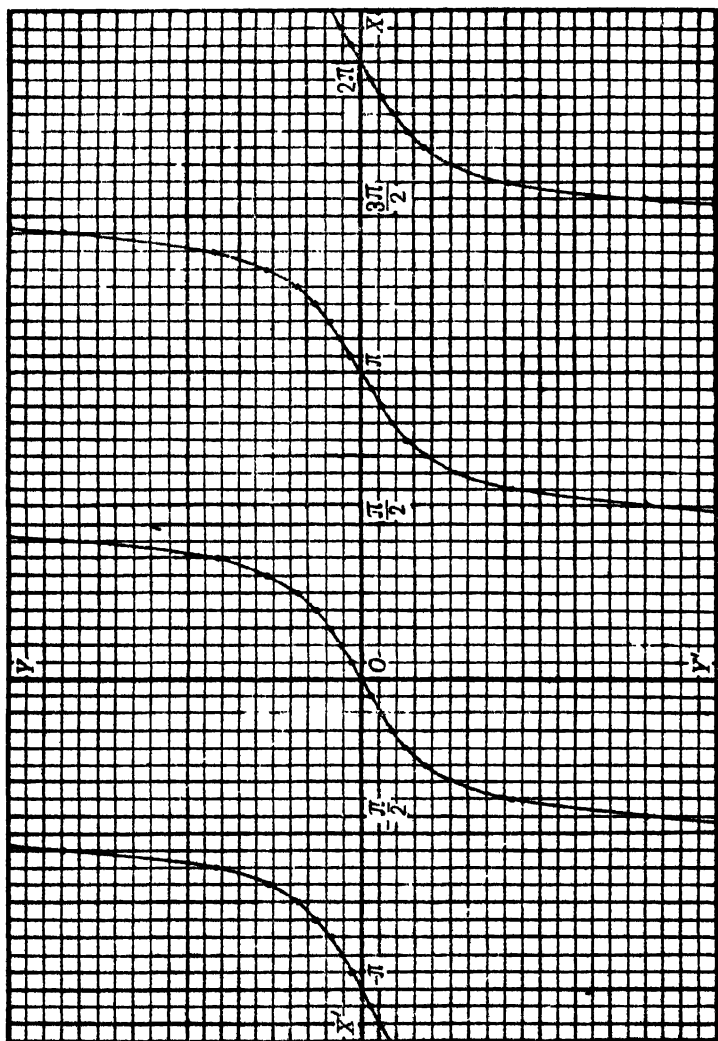
Tangent-graph

Note. *Peculiarities of the tangent-graph.*

From the figure, the following points will be apparent : (i) That the curve is not continuous, but consists of separate branches or portions, the points of discontinuities being the values of x corresponding to the odd multiples of $\frac{\pi}{2}$. (ii) As x passes through these points from the left to the right, the value of tan x suddenly changes from very large positive values on the left to very large negative values on the right. (iii) The lines parallel to y-axis corresponding to the odd multiple of $\frac{\pi}{2}$ are continuously approached by the graph on either side, but never actually meet. Such lines are called *asymptotes* to the curve. (iv) Since tan $(n\pi + x) =$ tan x, each branch is simply a repetition of the branch from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

107. Graph of cot x or *cotangent-graph.*

As before the values of x and y ($= \cot x$) are tabulated, and with the same scale as in the tangent-graph the points are plotted and joined free-hand

The graph is shown on the next page (between $x = -\pi$ to $x = +2\pi$).

This graph also, like the tangent-graph, is *discontinuous*, the points of discontinuity being $x = 0$ and $x = n\pi$. The portion between $x = 0$ and $x = \pi$ is repeated over and over again on either side, as is consequent from the formula cot $(n\pi + x) =$ cot x.

108. Graph of cosec x or *cosecant-graph.*

The corresponding values of x and y are tabulated at intervals of $10°$ of x as follows :—

	$-20°$	$-10°$	$0°$	$10°$	$20°$	$30°$	etc.	$80°$	$90°$	$100°$	$110°$	etc.
y or cosec x	$-2\cdot92$	$-5\cdot76$	∞	$5\cdot76$	$2\cdot92$	2	etc.	$1\cdot02$	1	$1\cdot02$	$1\cdot06$	etc.
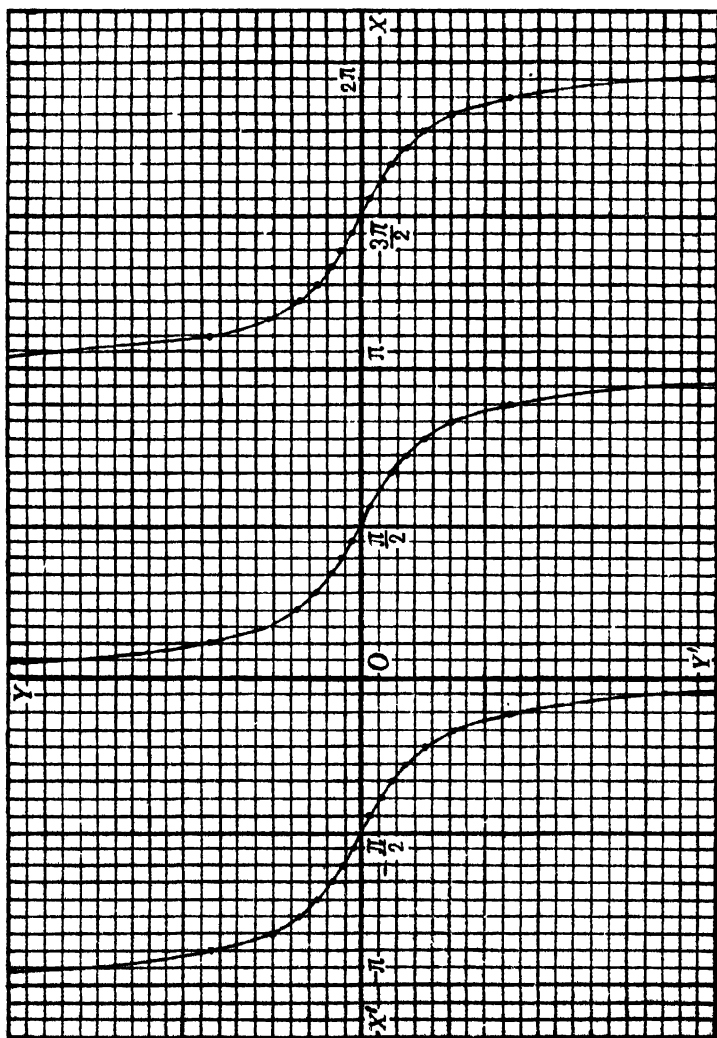
Cotangent-graph
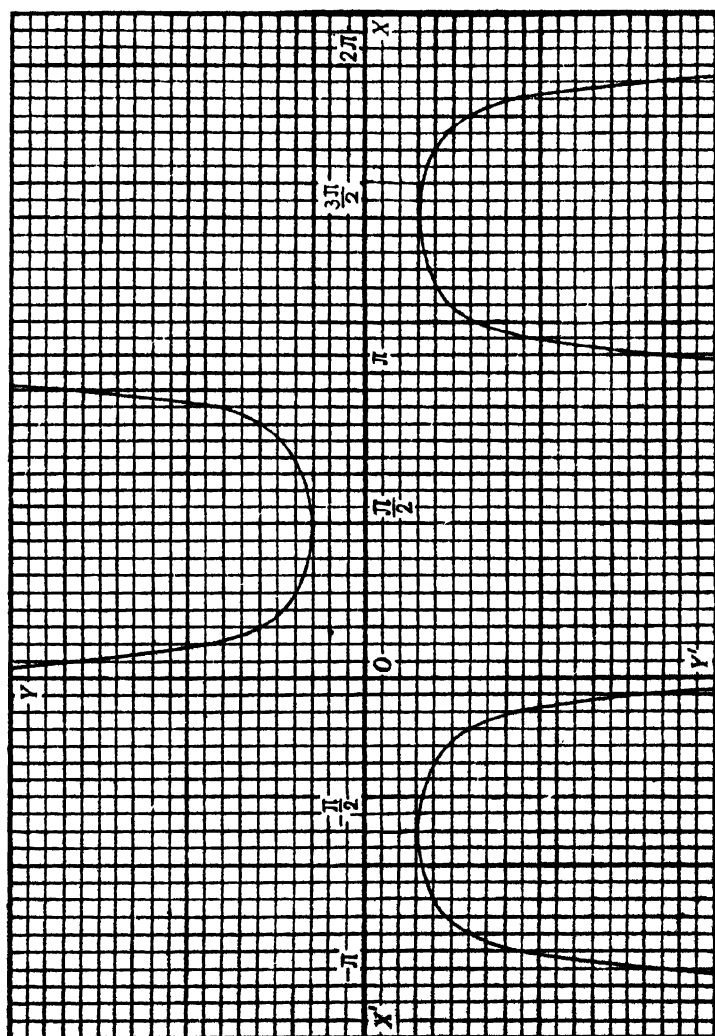
Cosecant-graph

[If the table of natural cosecants be not available, the table of natural sines may be used and the values of cosec x

$= \dfrac{1}{\sin x}$ may be calculated for different values of x.]

The scale is so chosen that 1 small division along OX represents $10°$, and 3 small divisions along OY represent unity.

The tabulated points are now plotted and joined free-hand.

The graph is shown on the previous page (between the range $x = -\pi$ to $x = 2\pi$).

Note. This graph also consists of *detached branches*, the points of discontinuity being $x=0$ and $x=n\pi$. The value of y never lies between ± 1, being always greater than 1 or less than -1. The lines $x=n\pi$ are asymptotes. The portion between $x=0$ to $x=2\pi$ is repeated on either side, over and over again.

109.   Graph of sec x or *secant-graph*.

The corresponding values of x and y ($= \sec x$) are tabulated as in the case of cosecant-graph, by making use of the table of cosines, if a table of secants be not available.

With the same scale as in the cosecant-graph, the tabulated points have been plotted and joined free-hand.

The graph is shown in the adjoining page (between the range $x = -\pi$ to $x = 2\pi$).

Note. It is apparent from the figure that the *secant-graph is exactly the same as the cosecant-graph, only shifted backwards (to the left) through a space of* $90°$.

This is due to the fact that cosec $(90° + x) = \sec x$.   [See *note below Art. 105*]
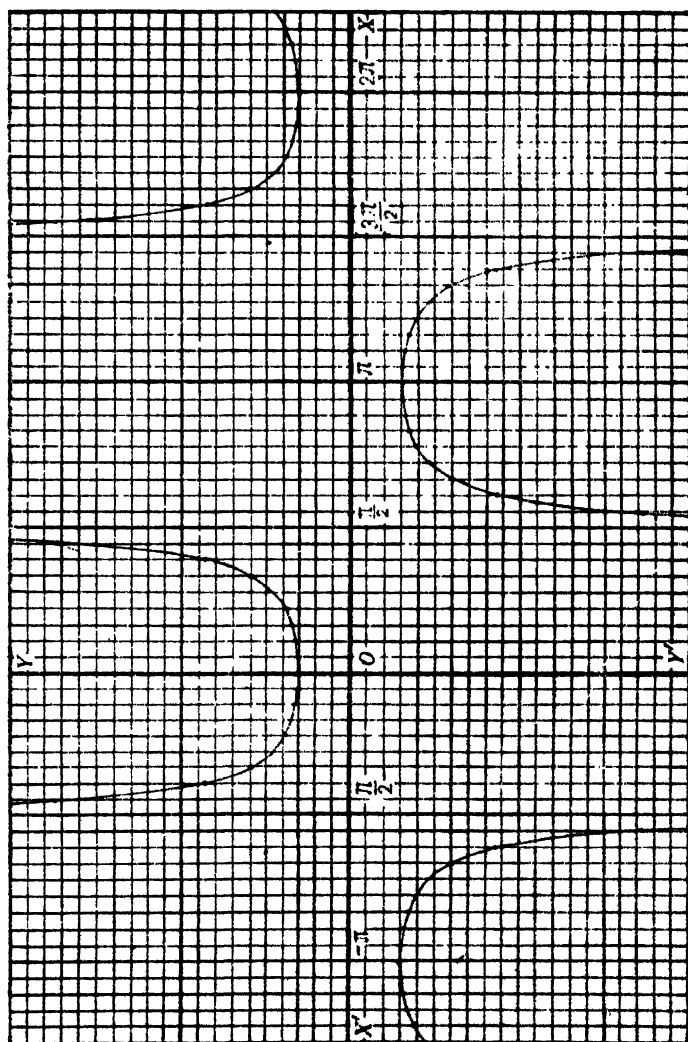
Secant-graph

110. Graphs of other Trigonometrical Expressions.

Graphs of other trigonometrical functions may be obtained in a similar manner.

We illustrate this by an example.

Ex. *Draw the graph of* $y = \sin x + \cos x$ *between the range* $x = 0$ *to* $x = 2\pi$, *and find from the graph the values of* x *for which* (i) $y = 0$, (ii) y *is maximum*, (iii) y *is minimum.*

[*C. U. 1934*]

From the table of natural sines and cosines, corresponding to each value of x, the values of $\sin x$ and $\cos x$ may be separately obtained and then added to give y ; or else we may write $y = \sin x + \cos x = \sqrt{2}\,(\sin x \cos \frac{1}{4}\pi + \cos x \sin \frac{1}{4}\pi)$ $= \sqrt{2}\sin (x + \frac{1}{4}\pi)$, and corresponding to any value of x, $\sin (x + \frac{1}{4}\pi)$ may be deduced from the sine-table, and this multiplied by $\sqrt{2}\,(= 1\text{·}414)$ will give y.

The corresponding values of x and y are tabulated at intervals of 10° of x, between the interval $x = 0$ to $x = 2\pi$ as follows :—

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°
y	1	1·15	1·27	1·37	1·41	1·41	1·37	1·27	1·15	1	·81

x	110°	120°	130°	140°	150°	160°	170°	180°	190°	200°
y	·59	·37	·13	− ·13	− ·37	− ·59	− ·81	−1	−1·15	−1·27
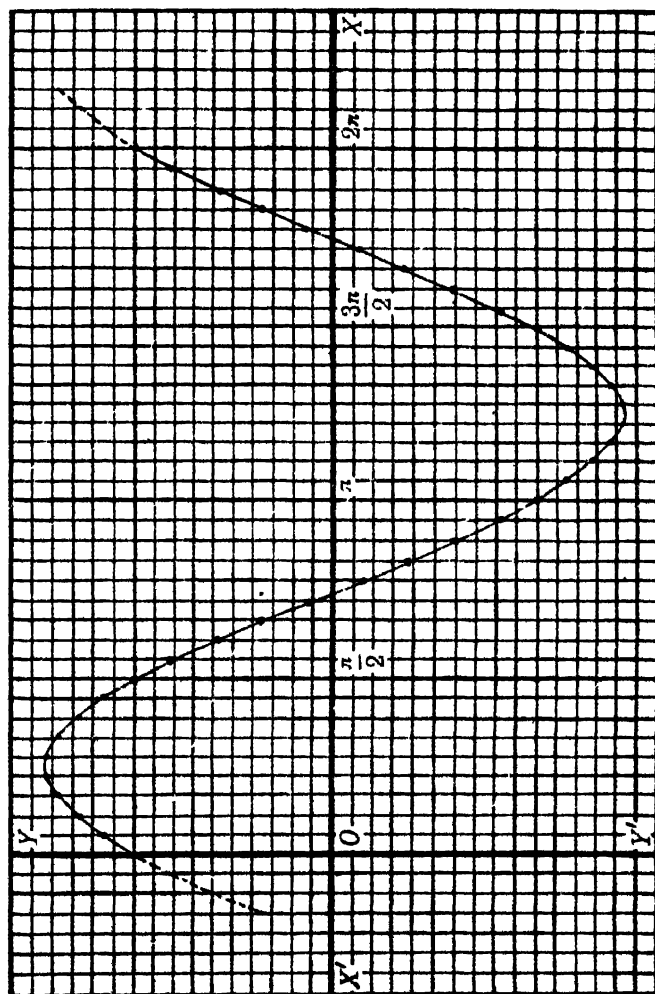
Graph of sin x + cos x

x	210°	220°	230°	240°	250°	260°	270°	280°
y	-1·37	-1·41	-1·41	-1·37	-1·27	-1·15	-1	-·81

x	290°	300°	310°	320°	330°	340°	350°	360°
y	-·59	-·37	-·13	·13	·37	·59	·81	1

The scale is chosen so that 1 small division along OX represents 10°, and 10 small divisions along OX represent unity.

The tabulated points are now plotted and joined. The graph is as shown on the previous page.

From the graph we find that (i) $y = 0$ when $x = 135°$, and $x = 315°$, (ii) y is maximum when $x = 45°$, (iii) y is minimum when $x = 225°$.

111. Graphical solution of Equations.

Trigonometrical equations, just like algebraic equations may be solved graphically. In fact in many practical cases, particularly where the solutions are not obtained in terms of the standard angles, the graphical method of solution is the only one which is found convenient and is actually adopted. The method is illustrated by the following examples.

Ex. 1. *Solve graphically the equation* $2 \sin^2 x = \cos 2x$,

giving only those solutions of x which lie between $-\dfrac{\pi}{2}$ *and* $\dfrac{3\pi}{2}$.

[*C. U. 1938, '46, '48*]

We draw two graphs, namely

$$y = 2 \sin^2 x = (1 - \cos 2x)$$

and $y = \cos 2x$

by tabulating the corresponding values of x and y for the two cases separately, making use of the table of natural
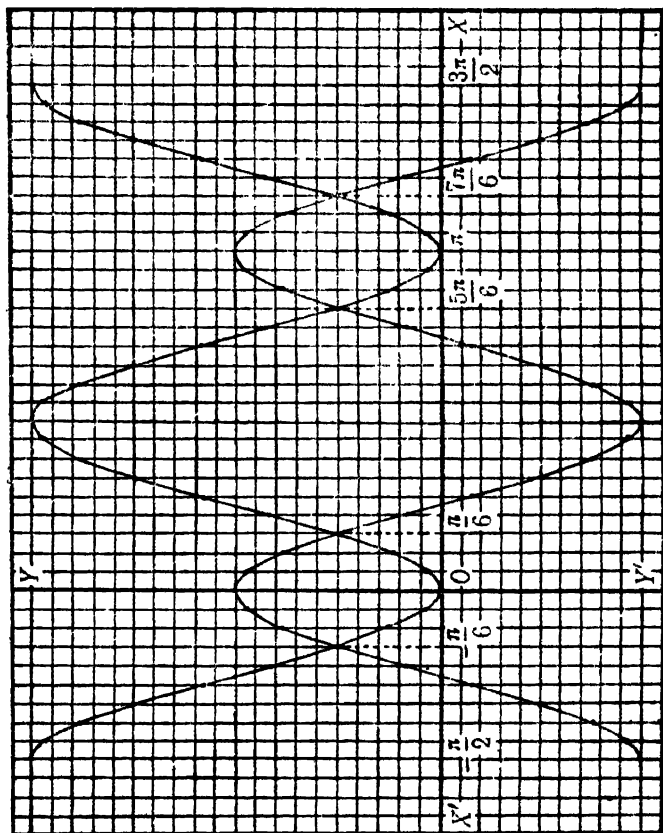
Graphical solution of $2 \sin^2 x = \cos 2x$.

cosines, for the range $x = -\dfrac{\pi}{2}$ to $x = \dfrac{3\pi}{2}$, at intervals of $10°$ or $15°$ of x.

With the same scale, namely, 1 small division along OX representing $10°$, and 10 small divisions along OY representing unity, we plot the tabulated values for the two cases in the same graph paper, and joining them, we get the two graphs, as shown in the adjoining page.

We find that the two graphs intersect, and thus have the same abscissæ and ordinates at the points for which

$$x = -\dfrac{\pi}{6}, \quad \dfrac{\pi}{6} \quad \dfrac{\pi}{6} \quad \dfrac{\pi}{6}$$

Thus, $2\sin^2 x = \cos 2x$ is satisfied for the values of x given by

$$x = -\dfrac{\pi}{6}, \; \dfrac{\pi}{6}, \; \dfrac{5\pi}{6} \text{ and } \dfrac{7\pi}{6}$$

which are the required solutions within the range

$$-\dfrac{\pi}{2} \text{ to } \dfrac{3\pi}{2}.$$

Ex. 2. *Solve graphically the equation* $\tan x = 2x$ *between* $x = 0$ *and* $x = \dfrac{\pi}{2}$.   [*C. U. 1939*]

Here, x is supposed to be measured in radians.

First of all we draw separately two graphs, namely

$$y = 2x \qquad \cdots \qquad \cdots \qquad \text{(i)}$$

and $\quad y = \tan x \qquad \cdots \qquad \cdots \qquad \text{(ii)}$
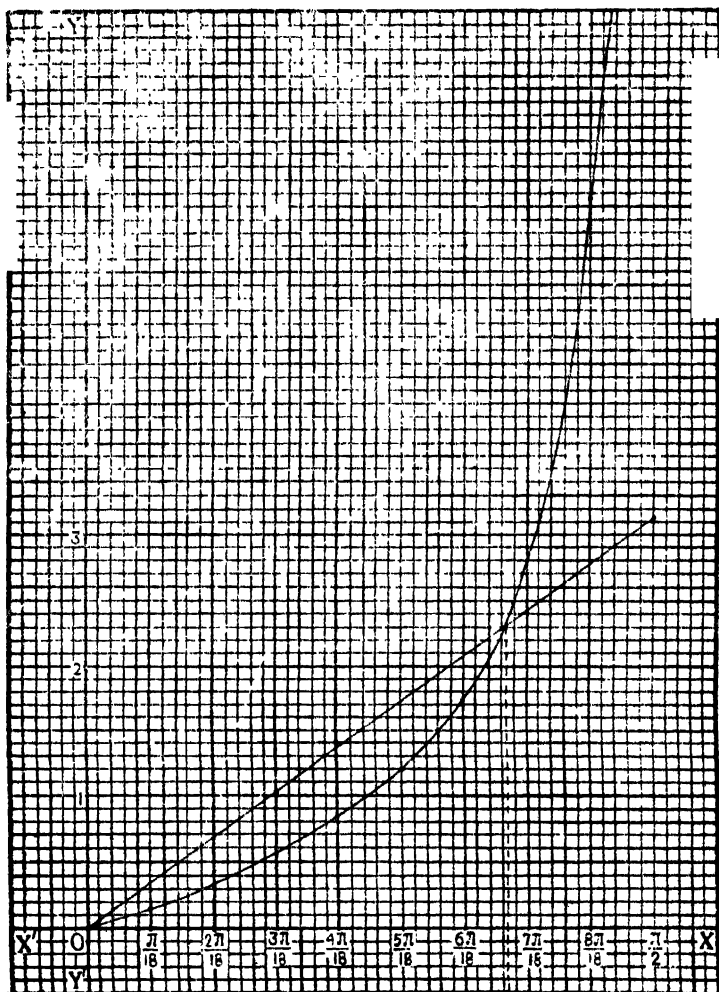
Graphical solution of tan $x = 2x$.

The corresponding values of x and y within the range $x = 0$ and $x = \dfrac{\pi}{2}$ are tabulated in case (i) as follows :

x (in radians)	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
y (i.e. $2x$) (numerical value)	0	1·05	2·10	3·15

and in case (ii) as follows :

x (in radians)	0	$\dfrac{\pi}{18}$	$\dfrac{2\pi}{18}$	$\dfrac{3\pi}{18}$	$\dfrac{4\pi}{18}$	$\dfrac{5\pi}{18}$	$\dfrac{6\pi}{18}$	$\dfrac{7\pi}{18}$	$\dfrac{8\pi}{18}$	$\dfrac{\pi}{2}$
y (i.e. $\tan x$) (numerical value)	0	·18	·36	·57	·84	1·19	1·73	2·75	5·67	∞

Now, choosing the same scale, namely 5 small divisions along OX to represent $\dfrac{\pi}{18}$ radians, and 10 small divisions along OY to represent unity, we plot the tabulated points for the two cases in the same graph paper and joining them we get the two graphs within the range $x = 0$ and $x = \dfrac{\pi}{2}$ as shown in the adjoining page.

We find that the two graphs intersect at the point given by $x = 0$ and also at the point corresponding to 33·5 small divisions along OX, which, from our chosen scale, represents $x = \dfrac{33·5}{5} \times \dfrac{\pi}{18}$ radians = 1·17 radians (approximately).

Hence, the given equation $\tan x = 2x$ is satisfied between $x = 0$ and $x = \dfrac{\pi}{2}$ by the values of x, namely $x = 0$ and $x = 1·17$ (approximately), which are the required solutions in radians.

Examples XVI

1. Draw the graphs of :

(i) sin $3x$ between $x = 0°$ to $x = 180°$.

(ii) tan $\frac{3}{4}x$ between $x = -\frac{1}{2}\pi$ to $x = \pi$.

(iii) sin θ cos θ between $\theta = -\pi$ to $\theta = +\pi$.

(iv) $\dfrac{1}{\cos^2\theta - \sin^2\theta}$ between $\theta = -\dfrac{\pi}{2}$ to $+\dfrac{\pi}{2}$.

(v) cos $(\pi \sin x)$ between $x = 0$ to $x = \frac{1}{2}\pi$.

(vi) sin $\theta - \sqrt{3}$ cos θ between $\theta = 0$ to $\theta = \pi$.

(vii) $\frac{1}{2}$ cosec $\frac{1}{2}x$ between $x = 0$ to $x = 2\pi$.

2. (i) Trace the changes in the sign of cos $\theta -$ sin θ as θ changes from $0°$ to $360°$. Verify your conclusions by a graph.

(ii) Trace the changes in sign and magnitude of $\dfrac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta}$.   [*B. H. U. 1931*]

3. Draw the graph of $y = \sin (x + \frac{1}{2}\pi)$ between the limits $x = -\pi$ and $x = +\pi$.

4. Draw the graphs of sin θ and cos θ between $\theta = 0$ and $\theta = \pi$. Find the points where the graphs intersect.

[*C. U. 1936, '46*]

5. Construct the graphs of tan x and cos x between 0 and $\frac{1}{2}\pi$ for x, making a tabulation of the values of y dividing the interval into 9 equal parts.

If tan $x = \cos x$, find approximately the value of x from the above two graphs.   [*C. U. 1943*]

6. Obtain graphically a solution of the equation tan $x = 1$, between $x = 0$ and $x = \frac{1}{2}\pi$.   [*C. U. 1937*]

[*Draw the graphs of $y = \tan x$ and $y = 1$.*]

7. Draw the graph of $\cos x - \sin 2x$ for values of x lying between $0°$ and $90°$ and hence obtain the least value of $\cos x - \sin 2x$ in this range.

8. Solve graphically the equations :

(i) $x - \tan x = 0$, between $x = 0$ and $x = \frac{1}{2}\pi$.

[C. U. 1945]

(ii) $5 \sin \theta + 2 \cos \theta = 5$, between $\theta = 0°$ and $\theta = 270°$.

[*Draw the graphs of $y = 5 \sin \theta + 2 \cos \theta$ and $y = 5$ and find the common points.*]   [C. U. 1947]

(iii) $\cot \theta - \tan \theta = 2$, between $\theta = 0$ and $\theta = \pi$.

[C. U. 1949]

(iv) $\mathrm{cosec}\ x = \cot x + \sqrt{3}$, between $x = 0$ and $x = \pi$.

(v) $\cos x = \sin 2x + \frac{1}{2}$, between $x = -\frac{1}{2}\pi$ and $x = +\frac{1}{2}\pi$.

(vi) $5 - \tan x = 2x$, between 0 and 2π.

(vii) $2 \sin x + x - 3 = 0$.

(viii) $x^2 = \cos x$.

(ix) $x = \cos^2 x$.

[*Draw the graphs of $y = \cos 2x$ and $y = 2x - 1$.*]

9. Represent by a graph the displacement given by

$$s = 2 \sin t + \sin 3t.$$

10. Show graphically that the equation $2 \sin x + \cos 2x = \frac{1}{2}x$ has only three real roots.

11. Sketch the graphs :

$y = x$, $y = \sin x$, $y = \tan x$, in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. From the nature of graphs near the origin, can you suggest any relation among them at the origin ?   [C. U. 1952]

MISCELLANEOUS THEOREMS AND EXAMPLES

Sec. A

HARDER PROBLEMS ON HEIGHTS AND DISTANCES

112. Some simple practical applications of Trigonometry, dealing with easy problems on determination of heights and distances, have already been discussed in Chapter V. The problems in the present section are of a more general character, requiring for their solutions, the general relations between the sides and angles of a triangle, as also some geometrical skill.

113. *To find the height and the distance of an inaccessible object standing on a horizontal plane.*
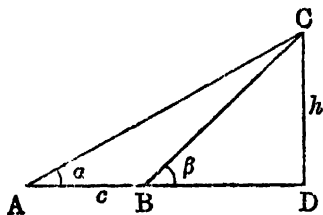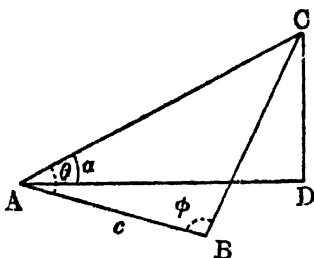
Fig. (i)   Fig. (ii)

Let CD be the object, which is observed from a point A on a horizontal ground, a being the observed elevation of its top C. Let h be the required height CD and d the required distance AD of the object from A.

Case I. If possible, measure off any suitable distance $AB\,(=c)$ from A directly towards the object, and from B let the observed elevation of C be β.

Then, from fig. (i),

$$c = AD - BD = h \cot a - h \cot \beta$$

$$= h \left(\frac{\cos a}{\sin a} - \frac{\cos \beta}{\sin \beta} \right) = \frac{h \sin (\beta - a)}{\sin a \sin \beta},$$

$\therefore \quad h = c \sin a \sin \beta \operatorname{cosec} (\beta - a).$

Also $d = AD = h \cot a = c \cos a \sin \beta \operatorname{cosec} (\beta - a).$

Note. Each of the above expressions for h and d is in a suitable form for logarithmic computation.

Case II. If however it is not convenient to measure the length AB directly towards the object, we may proceed as follows.

Measure off the length $AB(= c)$ in *any* suitable direction from A. From A let the observed elevation of C be a as before. The angles CAB and CBA are also observed from A and B respectively. Let these be θ and ϕ.

We get from fig. (ii) in this case,

in $\triangle ABC$, $\dfrac{AC}{\sin \phi} = \dfrac{AB}{\sin C}$,

i.e., $\quad = \dfrac{c}{\sin (180° - \theta - \phi)} = \dfrac{c}{\sin (\theta + \phi)}.$

$\therefore \quad AC = c \sin \phi \operatorname{cosec} (\theta + \phi).$

$\therefore \quad h = AC \sin a = c \sin a \sin \phi \operatorname{cosec} (\theta + \phi),$

and $\quad d = AD = AC \cos a = c \cos a \sin \phi \operatorname{cosec} (\theta + \phi).$

Note. Here also, the expressions for h and d are suitable for calculation by logarithm.

114. *To find the distance between two visible but inaccessible objects.*

Let P and Q be the objects whose distance apart is required.

Take two suitable points A and B for observation, the distance between which is measured, say c.
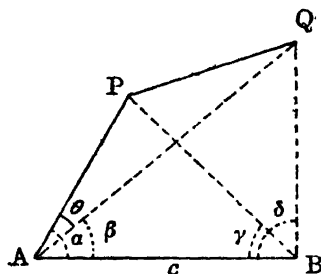
At A, measure the angles PAQ, PAB, and QAB (the second observation being unnecessary if all the four points P, A, B, Q are in the same plane, for in that case, $\angle PAB = \angle PAQ + \angle QAB$). Let these be θ, a and β respectively.

At B, measure the angles PBA and QBA, and let them be γ and δ.

From $\triangle PAB$, $\dfrac{PA}{\sin \gamma} = \dfrac{c}{\sin (180° - a - \gamma)} = \dfrac{c}{\sin (a + \gamma)}$, whence $PA = c \sin \gamma \operatorname{cosec} (a + \gamma)$.

Similarly, from $\triangle QAB$,

$$QA = c \sin \delta \operatorname{cosec} (\beta + \delta).$$

Lastly, from $\triangle PAQ$,

$$PQ^2 = PA^2 + QA^2 - 2PA \cdot QA \cdot \cos \theta.$$

Thus, PQ is determined.

115. Some more illustrative examples of harder problems on heights and distances are worked out below.

Ex. 1. *A flagstaff is fixed on the top of a tower standing on a horizontal plane. An observer walking directly towards the foot of the tower, observes the angle subtended*

by the flagstaff from two positions on his path to be the same namely 0. The distance between these two positions is d, and the angle subtended by the tower at his first position is a. Find the height of the tower and the length of the flagstaff.
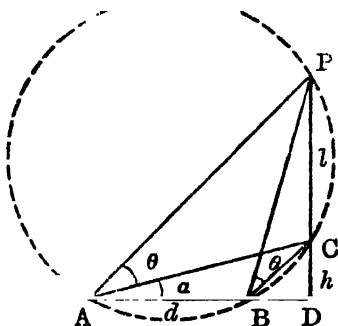
Let CD be the tower, PC the flagstaff, whose heights required are h and l respectively. A and B are the points of observation.

$\because\ \angle PAC = \angle PBC = \theta,$ the points P, A, B, C are concyclic,

$\therefore\ \angle CBD = \angle APC = 90° - \angle PAD = 90° - (\theta + a).$

Now, $\qquad d = AD - BD = h \cot a - h \cot (CBD)$

$\qquad\qquad = h \{\cot a - \tan (\theta + a)\}$

$\qquad\qquad = h \left\{ \dfrac{\cos a}{\sin a} - \dfrac{\sin (\theta + \beta)}{\cos (\theta + a)} \right\} = h \dfrac{\cos (\theta + 2a)}{\sin a \cos (\theta + a)}$

$\therefore\quad h = d \sin a \cos (\theta + a) \sec (\theta + 2a).$

Again, from $\triangle APC$,

$\qquad \dfrac{l}{\sin \theta} = \dfrac{AC}{\sin APC} = \dfrac{h}{\sin a \cos (\theta + a)} = \dfrac{d}{\cos (\theta + 2a)}.$

$\therefore\quad l = d \sin \theta \sec (\theta + 2a).$

Ex. 2. *A man walking towards a building, on which a flagstaff is fixed, observes the angle subtended by the flagstaff to be greatest, when he is at a distance d from the building. If θ be the observed greatest angle, find the length of the flagstaff and the height of the building.*
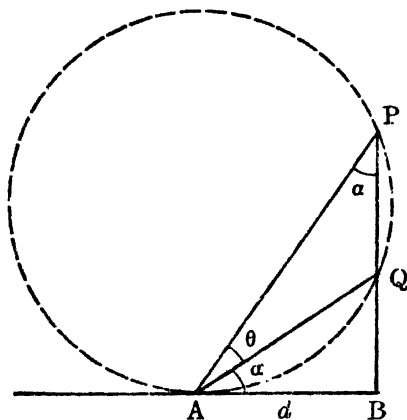
Let QB be the building, and PQ the flagstaff. It is easily seen from Geometry that the point of contact A of a circle through P and Q touching the path of the observer on the ground, is the point at which the angle subtended by PQ is the greatest.

Thus, $\angle QAB = \angle APQ = a$ say.

Then, $\angle PAB + \angle APB = 90°$

or, $\theta + 2a = 90°.$ ··· (i)

Now, $PQ = PB - QB = d \tan(\theta + a) - d \tan a$

$$= d \left\{ \frac{\sin(\theta + a)}{\cos(\theta + a)} - \frac{\sin a}{\cos a} \right\}$$

$$= d \frac{\sin \theta}{\cos(\theta + a)\cos a} = \frac{2d \sin \theta}{\cos(\theta + 2a) + \cos \theta}$$

$$= 2d \tan \theta.$$ [*from* (i)]

Also, $QB = d \tan a = d \tan\left(\frac{\pi}{4} - \frac{\theta}{2} \right).$

Ex. 3. *The angle of elevation of a light at the top of a distant tower from a point* 12 *ft. above a lake is* 24° 55′, *and the angle of depression of its reflection in the lake is* 35° 5′. *Find the height of the tower correct to two decimal places, having given*

$$log\ 2 = \text{·}30103, \qquad log\ 3 = \text{·}47712$$
$$log\ 588 = 2\text{·}76938, \qquad log\ 589 = 2\text{·}77012$$
$$L\ sin\ 10°\ 10' = 9\text{·}24677.$$
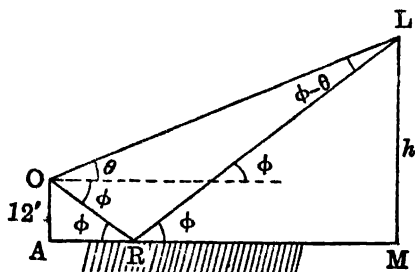
Let L be the light at the top of the tower LM, LRO the ray from L, which reflected in the lake at R, reaches the observer O, so that OR is the direction in which the reflexion is seen, and thus from the laws of reflexion, $\angle ORA = \angle LRM = \phi$ (say) which is evidently equal to the angle of depression of the reflexion, *i.e.*, 35° 5′.

Let θ denote the angle of elevation of L from O, *i.e.*, 24° 55′.

Now, from the figure, in $\triangle ORL$,

$$\frac{RL}{sin\ (\theta + \phi)} = \frac{OR}{sin\ (\phi - \theta)} = \frac{12}{sin\ \phi\ sin\ (\phi - \theta)}\ \text{ft.}$$

$$h = LM = RL\ sin\ \phi = 12\ \frac{sin\ (\theta + \phi)}{sin\ (\phi - \theta)} = 12\ \frac{sin\ 60°}{sin\ (10°\ 10')}$$

$$= \frac{6\ \sqrt{3}}{sin\ (10°\ 10')} = \frac{2.3^{\frac{3}{2}}}{sin\ (10°\ 10')}.$$

Hence, $\log h = \log (2.3^{\frac{3}{2}}) - \log \sin (10° \, 10')$

$\qquad = \log 2 + \tfrac{3}{2} \log 3 + 10 - L \sin (10° \, 10')$

$\qquad = ·30103 + \tfrac{3}{2} (·47712) + 10 - 9·24677$

$\qquad = 1·76994.$

From the given data, it is seen that

$\qquad \log h$ lies between $\log 58·8$ and $\log 58·9$.

Hence, if $h = 58·8 + x$, diff. for $·1 = 1·77012 - 1·76938$

$\qquad\qquad\qquad\qquad\qquad\qquad = ·00074,$

and diff. for $x = 1·76994 - 1·76938 = ·00056.$

$\therefore$ by the theory of proportional parts,

$\qquad \dfrac{x}{·1} = \dfrac{56}{74} = ·75 ; \quad \therefore \quad x = ·075 = ·08$ approximately.

Thus, $h = 58·88$ ft.

Examples XVII(a)

1. The angles of elevation of the top of a plam tree standing on horizontal ground, observed from two points A and B, distant 40 and 30 feet from the foot, and in the same straight line with it are found to be complementary. Prove that the height of the tree is nearly 35 feet, and that the angle subtended at the top of the tree by the line AB is $\sin^{-1} \tfrac{1}{7}$.

2. The angles of elevation of an aeroplane from two places one mile apart and from a point half way between them are found to be 60°, 30° and 45° respectively. Show that the height of the aeroplane $440 \sqrt{6}$ yards.

3. A building with ten storeys, each of equal height x ft., stands on one side of a wide street, and from a point on

the other side of the street directly opposite to the building, it is observed that the three uppermost storeys together and the two lowest storeys together subtend equal angles. Show that the width of the street is $x \sqrt{140}$ ft.

4. A two-storeyed building has the height of its lower storey 12 ft. and that of the upper storey 13 ft. Find at what distance the two storeys subtend equal angles to an observer's eye at a height 5 feet from the ground.

5. A vertical rod is erected in a horizontal rectangular field $ABCD$. The angular elevation of its top from A, B, C, D are a, β, γ, δ. Show that

$$\cot^2 a - \cot^2 \beta = \cot^2 \delta - \cot^2 \gamma.$$

6. The angles of elevation of a bird flying in a horizontal straight line, from a fixed point at four successive observations are a, β, γ, δ, the observations being taken at equal intervals of time. Assuming that the speed of the bird is uniform, show that

$$\cot^2 a - \cot^2 \delta = 3(\cot^2 \beta - \cot^2 \gamma).$$

7. A man on a hill observes that three towers on a horizontal plane subtend equal angles at his eye and that the angles of depression of their bases are a, β, γ. If a, b, c are the heights of the towers, prove that

$$\frac{\sin (\beta - \gamma)}{a \sin a} + \frac{\sin (\gamma - a)}{b \sin \beta} + \frac{\sin (a - \beta)}{c \sin \gamma} = 0.$$

8. A gun is fired from a fort F at a distance d from a station O, and from two stations A and B in a straight line with O and distant a and b respectively from O, the intervals between seeing the flash and hearing the report are t and t'. Show that the velocity of sound is

$$\sqrt{\frac{(d^2 - ab)(a - b)}{at'^2 - bt^2}}.$$

9. A person observes the elevation of the top of a telegraph post which is E. S. E. of him to be 45°, and at noon, the extremity of its shadow is to the N. E. of him ; if the length of the shadow be x, show that the height of the post is $x \sqrt{2 - \sqrt{2}}$.

10. A straight tree on the horizontal ground leans towards the North ; at two points due South and distant a, b respectively from the foot, the angular elevations of the top of the tree are a and β. Show that the inclination of the tree to the horizon is

$$\tan^{-1} \left(\frac{a-b}{a \cot \beta - \cot a} \right).$$

11. An observer on a carriage moving with a speed V along a straight road observes in one position that two distant trees are in the same line with him which is inclined at an angle θ to the road. After a time t, he observes that the trees subtend their greatest angle ϕ ; show that the distance between the tree is

$$2Vt \sin \theta \sin \phi / (\cos \theta + \cos \phi).$$

12. A train travelling on one of two straight intersecting railways subtends at a certain station on the other line, angles a and β, when the front of the first carriage and the end of the last carriage reach the junction respectively. Show that the angle of intersection of the two lines is

$$\tan^{-1} \frac{2 \sin a \sin \beta}{\sin (a \sim \beta)}.$$

13. Two vessels are sailing in parallel directions, and at one instant the bearing of one from the other $a°$ N. of E. After an hour's sailing the bearing of the first from the second is $\beta°$ N. of E., and after another hour the bearing is $\gamma°$ N. of E. Show that the vessels are sailing in a direction $\theta°$ N. of E., where

$$\sin (a - \theta) \sin (\gamma - \beta) = \sin (\beta - a) \sin (\gamma - \theta).$$

14. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground ; if the longest shadow it can cast is $3\frac{1}{3}$ times the length of the rod, calculate the altitude of the sun, having given

$$\log 3 = {\cdot}47712, \quad L \cos 72° \, 32' \, 30'' = 9{\cdot}47712.$$

15. A ship sailing N. E. is, at a particular moment, in a line with two light-houses, one of which is situated 5 miles

due N. of the other.   In 3 minutes and also in 21 minutes
the light-houses are found to subtend a right angle at the
ship.   Prove that the ship is sailing at the rate of 10 miles
an hour, and that the light-houses subtend their greatest
angle at the ship at the end of $3 \sqrt{7}$ minutes.

16.   A parachute was observed in the N. E. at the eleva-
tion 45° ; ten minutes afterwards it was found to be due N.
at an elevation $22\frac{1}{2}$°.   The parachute was descending at the
rate of 6 miles per hour, and was all along drifted uniformly
towards the West by the wind.   Show that wind blows at
the rate of 6 miles per hour.

17.   A person wishing to determine the height of a distant
temple observes the elevation of its top from a point on
the horizontal ground through its base to be 30°.   On
walking a distance $80 \sqrt{3}$ ft. in a certain direction, he finds
the elevation of the top to be the same as before, and then
on walking a distance 80 ft. at right angles to the former
direction, the elevation is found to be 45°.   Show that the
height of the temple is 80 ft.

18.   The shadow of a telegraph post is observed to be
half the known height of the post, and sometime afterwards
it is equal to the known height ; how much will the sun
have gone down interval, given   .

$$\log 2 = \text{'}30103, \ L \tan 63° \ 26' = 10\text{'}3009994$$

and diff. for $1' = 3159$.

19.   The side of a hill faces due S., and is inclined to
the horizon at an angle a.   A straight railway upon it is
inclined at an angle β to the horizon ; show that the bearing
of the railway is

$$\cos^{-1} (\cot a \tan \beta) \ \text{E. of N.}$$

20.   A spherical time-ball of diameter d at the top of
a tower subtends an angle $2a$ at a point on the ground from
which the elevation of its centre is θ ; prove that the height
of the centre of the ball above the ground is $\frac{1}{2}d \sin \theta \csc a$.

Sec. B—SUMMATION OF FINITE SERIES

116. Method of Difference.

When the rth term of a trigonometrical series can be expressed as the difference of two quantities, one of which is the same function of r as the other is of $(r+1)$, the sum of the series may be readily found as illustrated in the Examples 1 and 2 below.

Ex. 1. *Find the sum of the series :*

(i) $\operatorname{cosec} \theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 2^2\theta + \cdots + \operatorname{cosec} 2^{n-1}\theta.$

(ii) $\dfrac{\sin x}{\sin 2x \sin 3x} + \dfrac{\sin x}{\sin 3x \sin 4x} + \dfrac{\sin x}{\sin 4x \sin 5x}$

$\cdot$*to n terms.*

(i) We have $\operatorname{cosec} \theta = \dfrac{1}{\sin \theta} = \dfrac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta \sin \theta}$

$\qquad = \dfrac{\sin (\theta - \frac{1}{2}\theta)}{\sin \frac{1}{2}\theta \sin \theta}$

$\qquad = \dfrac{\sin \theta \cos \frac{1}{2}\theta - \cos \theta \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta \sin \theta}$

$\qquad = \cot \frac{1}{2}\theta - \cot \theta.$

Thus, $\qquad \operatorname{cosec} \theta = \cot \frac{1}{2}\theta - \cot \theta.$

Similarly, $\quad \operatorname{cosec} 2\theta = \cot \theta - \cot 2\theta.$

$\qquad \operatorname{cosec} 2^2\theta = \cot 2\theta - \cot 2^2\theta.$

$\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\qquad \operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta.$

$\therefore$ by addition, the required sum

$\qquad = \cot \frac{1}{2}\theta - \cot 2^{n-1}\theta.$

I. Tr.—15

(ii) Here, rth term

$$= \frac{\sin x}{\sin (r+1) x \sin (r+2) x}$$

$$= \frac{\sin \{(r+2) - (r+1)\} x}{\sin (r+1) x \sin (r+2) x}$$

$$= \frac{\sin (r+2)x \cos (r+1)x - \cos (r+2)x \sin (r+1)x}{\sin (r+1)x \sin (r+2)x}$$

$$= \cot (r+1) x - \cot (r+2) x.$$

Putting $r = 1, 2, 3, \ldots n$ and adding, the sum of the required series would be found to be

$$\cot 2x - \cot (n+2) x.$$

Ex. 2. *Find the sum of the series*

$$\tan^{-1} \frac{x}{1 + 1.2x^2} + \tan^{-1} \frac{x}{1 + 2.3x^2} + \cdots\cdots$$

$$+ \tan^{-1} \frac{x}{1 + n(n+1)x^2}$$

Here, rth term $= \tan^{-1} \dfrac{x}{1 + r(r+1)x^2}$

$$= \tan^{-1} \frac{(r+1)x - rx}{1 + (r+1)x.rx}$$

$$= \tan^{-1} (x+1)x - \tan^{-1} rx.$$

∴ putting $r = 1, 2, 3, \ldots\ldots n$, we have

$$\tan^{-1} \frac{x}{1 + 1.2x^2} = \tan^{-1} 2x - \tan^{-1} x$$

$$\tan^{-1} \frac{x}{1 + 2.3x^2} = \tan^{-1} 3x - \tan^{-1} 2x$$

$$\tan^{-1} \frac{x}{1 + n(n+1)x^2} = \tan^{-1} (n+1)x - \tan^{-1} nx.$$

by addition, the required sum

$$= \tan^{-1} (n+1)x - \tan^{-1} x.$$

117. Sometimes the *r*th term of a series, *when multiplied by a factor*, can be expressed as the difference of two quantities one of which is the same function of *r* as the other is of $(r + 1)$. It is illustrated in the following two cases.

(I) Sum of sines of n angles in A.P.

Let the angles in A.P. be a, $a + \beta$, $a + 2\beta$,... $\{a + (n - 1)\beta\}$, the first term being a, and the common difference, β.

Let S denote the sum of the series,

$$\sin a + \sin (a + \beta) + \sin (a + 2\beta) + \cdots + \sin \{a + (n - 1)\beta\}.$$

Multiplying each term of the above series by

2 *sin (half difference)* i.e., by $2 \sin \tfrac{1}{2}\beta$, we have

$$2 \sin a \sin \tfrac{1}{2}\beta = \cos (a - \tfrac{1}{2}\beta) - \cos (a + \tfrac{1}{2}\beta)$$
$$2 \sin (a + \beta) \sin \tfrac{1}{2}\beta = \cos (a + \tfrac{1}{2}\beta) - \cos (a + \tfrac{3}{2}\beta)$$
$$2 \sin (a + 2\beta) \sin \tfrac{1}{2}\beta = \cos (a + \tfrac{3}{2}\beta) - \cos (a + \tfrac{5}{2}\beta).$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2 \sin \{a + (n - 1)\beta\} \sin \tfrac{1}{2}\beta$$
$$= \cos \left(a + \frac{2n - 3}{2}\beta\right) - \cos \left(a + \frac{2n - 1}{2}\beta\right).$$

Adding vertically, we have

$$2 \sin \tfrac{1}{2}\beta \cdot S = \cos\left(a - \frac{\beta}{2}\right) - \cos\left(a + \frac{(2n - 1)}{2}\beta\right)$$

$$= 2 \sin \left(a + \frac{n - 1}{2}\beta\right) \sin \frac{n\beta}{2}.$$

$$\therefore \qquad S = \frac{\sin \dfrac{n\beta}{2}}{\sin \dfrac{\beta}{2}} \sin \left(a + \frac{n - 1}{2}\beta\right).$$

Cor. Putting $\beta = a$, we get

$$\sin a + \sin 2a + \sin 3a + \cdots + \sin na$$

$$= \frac{\sin \dfrac{na}{2}}{\sin \dfrac{a}{2}} \sin \frac{n + 1}{2} a$$

(II) Sum of cosines of n angles in A.P.

As before, let S denote the sum of the series

$$\cos u + \cos(a+\beta) + \cos(a+2\beta) + \cdots + \cos\{a+(n-1)\beta\}.$$

Multiplying each term of the above series by

$2\ sin\ (half\ difference)$, we have

$$2\cos a \cdot \sin\tfrac{1}{2}\beta = \sin(a+\tfrac{1}{2}\beta) - \sin(a-\tfrac{1}{2}\beta)$$

$$2\cos(a+\beta) \cdot \sin\tfrac{1}{2}\beta = \sin(a+\tfrac{3}{2}\beta) - \sin(a+\tfrac{1}{2}\beta)$$

$$2\cos(a+2\beta) \cdot \sin\tfrac{1}{2}\beta = \sin(a+\tfrac{5}{2}\beta) - \sin(a+\tfrac{3}{2}\beta)$$

....

$$2\cos\{a+(n-1)\beta\} \cdot \sin\tfrac{1}{2}\beta$$
$$= \sin\left(a+\frac{2n-1}{2}\beta\right) - \sin\left(a+\frac{2n-3}{2}\beta\right).$$

Adding vertically, we have

$$2\sin\tfrac{1}{2}\beta . S = \sin\left(a+\frac{2n-1}{2}\beta\right) - \sin\left(a-\frac{\beta}{2}\right)$$

$$= 2\cos\left(a+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}.$$

$$\therefore \qquad S = \frac{\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}}\cdot\cos\left(a+\frac{n-1}{2}\beta\right).$$

Cor.   Putting $\beta = a$, we get

$$\cos a + \cos 2a + \cos 3a + \cdots + \cos na = \frac{\sin\dfrac{na}{2}}{\sin\dfrac{a}{2}}\cos\frac{n+1}{2}a.$$

Note.   The sum of the cosine series may be deduced from that of the sine series by writing $a+\dfrac{\pi}{2}$ for a.

As an aid to memory, the two formulæ of this article may be expressed in language as follows :

since, $a + \dfrac{n-1}{2}\beta = \dfrac{a + a + (n-1)\beta}{2}$,

∴ **Sum of sines of n angles in A.P.**

$$= \dfrac{\sin \dfrac{n.diff.}{2}}{\sin \dfrac{diff.}{2}} \sin \dfrac{first\ angle + last\ angle}{2}.$$

Sum of cosines of n angles in A.P.

$$= \dfrac{\sin \dfrac{n.diff.}{2}}{\sin \dfrac{diff.}{2}} \cos \dfrac{first\ angle + last\ angle}{2}.$$

Note. From the above formulæ, it is clear that if $\sin \dfrac{n\beta}{2} = 0$, then the sum of the sine series as also the sum of the cosine series is zero.

Now, if $\sin \dfrac{n\beta}{2} = 0$, then $\dfrac{n\beta}{2} = k\pi$, or $,\beta = \dfrac{2k\pi}{n}$, where k is an integer.

Thus, *the sum of the sines and the sum of the cosines of n angles in A.P. are each equal to serc when the common difference of the angles is an even multiple of* $\dfrac{\pi}{n}$.

Ex. 1. *Find the sum of n terms of the series*

$$\sin a - \sin (a + \beta) + \sin (a + 2\beta) - \cdots\cdots$$

Since, $\sin (\pi + \theta) = -\sin \theta$; $\sin (2\pi + \theta) = \sin \theta$ etc.

∴ the series is equal to

$$\sin a + \sin (\pi + a + \beta) + \sin (2\pi + a + 2\beta) + \cdots\cdots,$$

i.e., equal to a series in which the common difference of the angles is $\beta + \pi$ and the last angle is $a + (n-1)(\beta + \pi)$.

$$S = \dfrac{\sin \dfrac{n(\beta + \pi)}{2}}{\sin \dfrac{\beta + \pi}{2}} \sin \left\{ a + \dfrac{(n-1)(\beta + \pi)}{2} \right\}.$$

Ex. 2. *Find the sum of the series*

$$sin^2\theta + sin^2 2\theta + sin^2 3\theta + \cdots + sin^2 n\theta.$$

Since, $sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, $sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$, etc.

∴ the given series

$$= \frac{1}{2}(1 - \cos 2\theta) + \frac{1}{2}(1 - \cos 4\theta) + \cdots + \frac{1}{2}(1 - \cos 2n\theta)$$

$$= \frac{n}{2} - \frac{1}{2}(\cos 2\theta + \cos 4\theta + \cdots + \cos 2n\theta)$$

$$= \frac{n}{2} - \frac{1}{2}\frac{\sin n\theta}{\sin \theta}\cos(n+1)\theta. \quad [\text{ by } \textit{Art. 117}]$$

Ex. 3. *Sum the series*

$$\cos \alpha + 2\cos(\alpha + \beta) + 3\cos(\alpha + 2\beta) + \cdots$$
$$\cdots + n\cos\{\alpha + (n-1)\beta\}.$$

Let u_r denote the rth term and S denote the sum of the given series.

Now, $2\cos\beta.u_r = 2\cos\beta . r\cos\{\alpha + (r-1)\beta\}$

$$= r[\cos(\alpha + r\beta) + \cos\{\alpha + (r-2)\beta\}]. .$$

∴ putting $r = 1, 2, 3,\ldots n$ and adding together,

we get $2\cos\beta . S$.

Now, subtract $2\cos\beta . S$ from $2S$; then

$$2S(1 - \cos\beta) = (n+1)\cos\{\alpha + (n-1)\beta\}$$
$$- \cos(\alpha - \beta) - n\cos(\alpha + n\beta).$$

Then, dividing by $2(1 - \cos\beta)$, S, the sum of the required series would be obtained.

Note. Similarly the sum of the series

$$\sin\alpha + 2\sin(\alpha + \beta) + 3\sin(\alpha + 2\beta) + \cdots + n\sin\{\alpha + (n-1)\beta\}$$

would be obtained.

Examples XVII(b)

Sum the following series to n terms (*Ex. 1 to 10*):

1. $\sin a + \sin\left(a - \dfrac{\pi}{n}\right) + \sin\left(a - \dfrac{2\pi}{n}\right) + \cdots\cdots$

2. $\cos a + \cos\left(a + \dfrac{2\pi}{n}\right) + \cos\left(a + \dfrac{4\pi}{n}\right) + \cdots\cdots$

3. $\sin a - \sin 2a + \sin 3a - \cdots\cdots$

4. $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \cdots\cdots$

5. $\sin^3 a + \sin^3 3a + \sin^3 5a + \cdots\cdots$

6. $\sin^2\theta - \sin^2 2\theta + \sin^2 3\theta - \sin^2 4\theta + \cdots\cdots$

7. $\sin^4 a + \sin^4 2a + \sin^4 3a + \cdots\cdots$

8. $\cos\theta - \sin 2\theta - \cos 3\theta + \sin 4\theta + \cos 5\theta - \sin 6\theta - \cdots\cdots$

9. $\sin a \sin 2a + \sin 2a \sin 3a + \sin 3a \sin 4a + \cdots\cdots$

10. $\cos a \cos 3a + \cos 3a \cos 5a + \cos 5a \cos 7a + \cdots\cdots$

Find the sum of the following series (*Ex. 11 to 14*):

11. $\cos\dfrac{\pi}{19} + \cos\dfrac{3\pi}{19} + \cos\dfrac{5\pi}{19} + \cdots + \cos\dfrac{17\pi}{19}$.

12. $\sin 5° + \sin 77° + \sin 149° + \cdots\cdots + \sin 293°$.

13. $\sin\dfrac{2\pi}{n} + \sin\dfrac{4\pi}{n} + \sin\dfrac{6\pi}{n} + \cdots + \sin\dfrac{2n\pi}{n}$.

14. $\sin na + \sin(n-1)a + \sin(n-2)a + \cdots$ to $2n$ terms.

15. Prove that

(i) $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \cdots \text{ to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \cdots \text{ to } n \text{ terms}} = \tan n\theta$.

(ii) $\sin^2 a + \sin^2\left(a + \dfrac{2\pi}{n}\right) + \sin^2\left(a + \dfrac{4\pi}{n}\right) + \cdots$ to n terms
$$= \tfrac{1}{2}n.$$

Sum to n terms (*Ex. 16 to 28*) :

16. $\sec a \sec 2a + \sec 2a \sec 3a + \sec 3a \sec 4a + \cdots$

17. $\dfrac{1}{\sin \theta \sin 2\theta} + \dfrac{1}{\sin 2\theta \sin 3\theta} + \dfrac{1}{\sin 3\theta \sin 4\theta} + \cdots$

18. $\dfrac{1}{\cos a + \cos 3a} + \dfrac{1}{\cos a + \cos 5a} + \dfrac{1}{\cos a + \cos 7a} + \cdots$

19. $\cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + \cot 3\theta \cot 4\theta + \cdot$

20. $\tan a + 2 \tan 2a + 4 \tan 4a + 8 \tan 8a + \cdots\cdots$

$$[\ \tan a = \cot a - 2 \cot 2a\ ]$$

21. $\sin 2\theta \sin^2 \cdot \dfrac{2\theta}{2} + \sin 3\theta \sin^2 \dfrac{3\theta}{2} + \sin 4\theta \sin^2 \dfrac{4\theta}{2} + \cdot$

22. $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 3^2 x} + \dfrac{\sin 3^2 x}{\cos 3^3 x} + \cdots\cdots$

$$[\ \textit{1st term} = \tfrac{1}{2}\ (\tan 3x - \tan x)\ ]$$

23. $\tan^{-1} \dfrac{1}{1 + 1 + 1^2} + \tan^{-1} \dfrac{1}{1 + 2 + 2^2}$

$$+ \tan^{-1} \dfrac{1}{1 + 3 + 3^2} + \cdots\cdots$$

24. $\tan^{-1} \dfrac{2}{1 + 1.3} + \tan^{-1} \dfrac{2}{1 + 3.5} + \tan^{-1} \dfrac{2}{1 + 5.7} + \cdots\cdots$

25. $\cot^{-1} (2.1^2) + \cot^{-1} (2.2^2) + \cot^{-1} (2.3^2) + \cdots$

26. $\tan x + \dfrac{1}{2} \tan \dfrac{1}{2} x + \dfrac{1}{2^2} \tan \dfrac{1}{2^2} x + \cdots\cdots$

27. $\cos x \cos 2x \cos 3x + \cos 2x \cos 3x \cos 4x + \cdot$

28. $\cos \theta + 2 \cos 2\theta + 3 \cos 3\theta + \cdots + n \cos n\theta.$

29.   Find the sum of the series :

(i) $\sin a + \sin 2a + \sin 3a + \cdots$ to n terms

and   (ii) $\sin a + \sin 3a + \sin 5a + \cdots$ to n terms

and hence deduce respectively the sums of the series

    (a) $1 + 2 + 3 + \cdots$ to n terms

and (b) $1 + 3 + 5 + \cdots$ to n terms.

30. Sum the series

    $\tan x \tan 2x + \tan 2x \tan 3x + \cdots\cdots + \tan nx \tan (n+1)x$

and hence deduce the sum of the series

        $1.2 + 2.3 + \cdots + n(n+1)$.

31. If β be the exterior angle of a regular polygon of n sides, show that

    (i) $\sin a + \sin (a + \beta) + \sin (a + 2\beta) + \cdots$ to n terms $= 0$.

    (ii) $\cos a + \cos (a + \beta) + \cos (a + 2\beta) + \cdots$ to n terms $= 0$.

32. A regular polygon of n sides is inscribed in a circle of radius a; prove that

    (i) the sum of the lengths of the perpendiculars drawn from the angular points upon any diameter is zero;

    (ii) the sum of the lengths of the lines joining any vertex to each of the other vertices is $2a \cot \dfrac{\pi}{2n}$.

Sec. C—ELIMINATION

118. The elimination of trigonometrical functions from given equations is a very important and common mathematical problem. There are no set rules to effect the elimination. The form of the equations will often suggest special methods, and in addition to the usual algebraical artifices, we shall always have at our disposal the identical relations existing among the trigonometrical functions.

    The following examples will illustrate some useful methods of elimination.

Ex. 1. *Eliminate θ from the equations*

$$a \cos \theta + b \sin \theta + c = 0$$
$$a' \cos \theta + b' \sin \theta + c' = 0.$$

From the given equations, we have by cross-multiplication,

$$\frac{\cos \theta}{bc' - b'c} = \frac{\sin \theta}{ca' - c'a} = \frac{1}{ab' - a'b}.$$

$$\therefore \quad \cos \theta = \frac{bc' - b'c}{ab' - a'b}, \text{ and } \sin \theta = \frac{ca' - c'a}{ab' - a'b}.$$

Squaring and adding, we get

$$(bc' - b'c)^2 + (ca' - c'a)^2 = (ab' - a'b)^2$$

as the required eliminant.

Ex. 2. *Eliminate θ from the equations*

$$x \sin \theta + y \cos \theta = 2a \sin 2\theta$$
$$x \cos \theta - y \sin \theta = a \cos 2\theta.$$

Solving as simultaneous equations in x and y, we have

$$x = a(\cos 2\theta \cos \theta + 2 \sin 2\theta \sin \theta)$$
$$= a[\cos (2\theta - \theta) + \sin 2\theta \sin \theta]$$
$$= a(\cos \theta + 2 \sin^2\theta \cos \theta) ;$$
$$y = a(2 \sin 2\theta \cos \theta - \cos 2\theta \sin \theta)$$
$$= a(\sin \theta + \sin 2\theta \cos \theta) = a(\sin \theta + 2 \sin \theta \cos^2\theta).$$

$$\therefore \quad x + y = a(\sin \theta + \cos \theta)(1 + 2 \sin \theta \cos \theta)$$
$$= a(\sin \theta + \cos \theta)(\sin \theta + \cos \theta)^2 = a(\cos \theta + \sin \theta)^3.$$

Similarly,

$$x - y = a(\cos \theta - \sin \theta)(1 - 2 \sin \theta \cos \theta)$$
$$= a(\cos \theta - \sin \theta)^3.$$

$$\therefore \quad a^{\frac{1}{3}}(\cos \theta + \sin \theta) = (x+y)^{\frac{1}{3}} \qquad \cdots \quad \text{(i)}$$

$$a^{\frac{1}{3}}(\cos \theta - \sin \theta) = (x-y)^{\frac{1}{3}}. \qquad \cdots \quad \text{(ii)}$$

Hence, squaring and adding (i) and (ii), we have,

$$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

as the required eliminant.

Ex. 3. *Eliminate x and y from the equations*

$$a \sin^2 x + b \cos^2 x = c, \ b \sin^2 y + a \cos^2 y = d,$$
$$a \tan x = b \tan y.$$

From the first equation, we have

$$a \sin^2 x + b \cos^2 x = c(\sin^2 x + \cos^2 x).$$

$$\therefore \quad (a-c)\sin^2 x = (c-b)\cos^2 x.$$

$$\therefore \quad \tan^2 x = \frac{c-b}{a-c}.$$

From the second equation, we have similarly

$$b \sin^2 y + a \cos^2 y = d(\sin^2 y + \cos^2 y).$$

$$\therefore \quad \tan^2 y = \frac{d-a}{b-d}.$$

From the third equation,

$$a^2 \tan^2 x = b^2 \tan^2 y.$$

$$\therefore \quad \frac{a^2(c-b)}{a-c} = \frac{b^2(d-a)}{b-d}.$$

This, when simplified, reduces to

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d},$$ the required eliminant.

Examples XVII(c)

Eliminate θ from the following pair of equations (*Ex. 1 to 15*) :

1. $\cot \theta (1 + \sin \theta) = 4a.$

$\cot \theta (1 - \sin \theta) = 4b.$

2. $x = a \cos \theta + b \cos 2\theta.$

$y = a \sin \theta + b \sin 2\theta.$

3. $x = \tan \theta + \tan 2\theta.$

$y = \cot \theta + \cot 2\theta.$

4. $a \sin \theta + b \cos \theta = 1.$

$a \operatorname{cosec} \theta - b \sec \theta = 1.$

5. $x = \sin \theta + \cos \theta \sin 2\theta$

$y = \cos \theta + \sin \theta \sin 2\theta.$

6. $x + a = a(2 \cos \theta - \cos 2\theta)$

$y = a(2 \sin \theta - \sin 2\theta).$

7. $x = 3 \sin \theta - \sin 3\theta$

$y = \cos 3\theta + 3 \cos \theta.$

8. $x \cot \theta + \tan \theta$

$y = \sec \theta - \cos \theta.$

9. $x \sin \theta - y \cos \theta = \sqrt{x^2 + y^2}$

$\dfrac{\cos^2 \theta}{a^2} + \dfrac{\sin^2 \theta}{b^2} = \dfrac{1}{x^2 + y^2}.$

10. $\dfrac{x}{a} = \cos \theta + \cos 2\theta$

$\dfrac{y}{b} = \sin \theta + \sin 2\theta.$

11. $\dfrac{ax}{\cos \theta} - \dfrac{by}{\sin \theta} = a^2 - b^2$

$\dfrac{ax \sin \theta}{\cos^2 \theta} + \dfrac{by \cos \theta}{\sin^2 \theta} = 0.$

12. $\dfrac{x}{a} \cos \theta - \dfrac{y}{b} \sin \theta = \cos 2\theta$

$\dfrac{x}{a} \sin \theta + \dfrac{y}{b} \cos \theta = 2 \sin 2\theta.$

13. $x = \operatorname{cosec} \theta - \sin \theta$

$y = \sec \theta - \cos \theta.$

14. $\sin \theta + \cos \theta = a$

$\sin^3 \theta + \cos^3 \theta = b.$

15. $\dfrac{x}{a} \cos \theta + \dfrac{y}{b} \sin \theta = 1$

$x \sin \theta - y \cos \theta = (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{1}{2}}.$

Eliminate θ and ϕ from the following equations (*Ex. 16 to 19*) :—

16. $\sin \theta + \sin \phi = x,\ \cos \theta + \cos \phi = y,\ \theta - \phi = a.$

17. $\tan \theta + \tan \phi = a,\ \cot \theta + \cot \phi = b,\ \theta + \phi = a.$

18. $a \sin^2 \theta + b \cos^2 \theta = a \cos^2 \phi + b \sin^2 \phi = 1,$

$a \tan \theta = b \tan \phi.$

19. $\sin \theta + \sin \phi = a,\ \cos \theta + \cos \phi = b,\ \sin 2\theta + \sin 2\phi = 2c.$

20. If $(a + b) \tan (\theta - \phi) = (a - b) \tan (\theta + \phi)$ and

$a \cos 2\phi + b \cos 2\theta = c,$ show that

$a^2 - b^2 + c^2 = 2ac \cos 2\phi.$

APPENDIX

1. *To prove that*

$$\sin \theta < \theta < \tan \theta,$$

where θ is the circular measure of any positive acute angle.
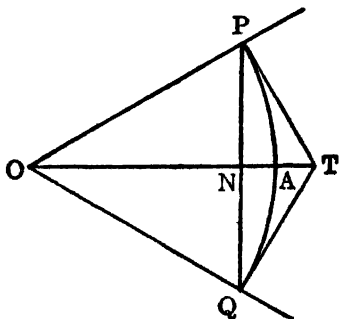
Let AOP be a positive acute angle whose radian measure is θ, and let QOA be an equal angle on the other side of OA. With centre O and any radius, a circle is drawn cutting OP, OA, OQ at P, A, Q respectively. PQ is joined cutting OA at N. The triangles OPN and OQN are easily seen to be congruent, so that $PN = QN$ and PNQ is perpendicular to OA. The tangent PT to the circle at P cuts OA at T, therefore $\angle OPT$ is a right angle. TQ being joined, the triangles OPT and OQT are easily proved to be congruent, so that $TP = TQ$.

The figure is thus symmetrical about OA.

Then, from the figure,

$$\sin \theta = \frac{PN}{OP} = \frac{1}{2} \cdot \frac{PQ}{OP}$$

$$\theta = \frac{\text{arc } PA}{OP} = \frac{1}{2} \cdot \frac{\text{arc } PAQ}{OP}$$

$$\tan \theta = \frac{PT}{OP} = \frac{1}{2} \cdot \frac{PT + QT}{OP}.$$

Now, we may take it as axiomatic that the straight line PQ is less than the curved arc PAQ, and that the curved arc PAQ which always bends the same way, being within the triangle PTQ, is less than $PT + TQ$.

Hence, since $PQ <$ arc $PAQ < PT + QT$,

we have, on dividing throughout by $2OP$,

$$\sin \theta < \theta < \tan \theta.$$

Alternative method :

Let ABC be a circle whose centre is O and radius r,
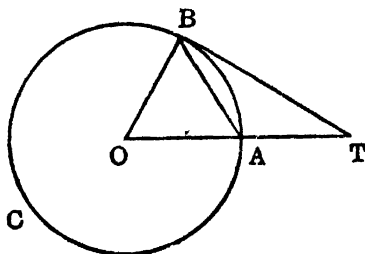
Let $AOB = \theta$ radians.

Draw BT the tangent at B to meet OA produced at T ; then $BT = r \tan \theta$.

We know that if the angle of a sector of a circle of radius r is θ radians, its area $= \frac{1}{2}r^2\theta$.

Now, from the figure it is obvious that

$$\triangle OAB < \text{sector } OAB < \triangle OBT.$$

$\therefore \quad \frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2\theta < \frac{1}{2}r.r \tan \theta,$

i.e., $\quad \sin \theta < \theta < \tan \theta.$

Cor. If now θ becomes infinitely small, we can prove

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin \theta}{\theta} = 1,$$

$$\underset{\theta \to 0}{\text{Lt}} \cos \theta = 1,$$

and $$\underset{\theta \to 0}{\text{Lt}} \frac{\tan \theta}{\theta} = 1.$$

For, since, $\sin \theta < \theta < \tan \theta$,

we get, by dividing by $\sin \theta$,

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

This is true, however small θ may be, provided it is positive. When θ becomes infinitely small, OP and ON practically come into coincidence, so that

$$\cos \theta = \frac{ON}{OP} \text{ ultimately becomes 1.}$$

Hence, $\underset{\theta \to 0}{Lt} \cos \theta = 1$.

In that case $\frac{1}{\cos \theta}$ also tends to the value 1. But $\frac{\theta}{\sin \theta}$ always lies between 1 and $\frac{1}{\cos \theta}$ which ultimately come into coincidence, and so $\frac{\theta}{\sin \theta}$ also ultimately coincides with 1.

Thus, $\frac{\sin \theta}{\theta} = 1$ in the limit.

Again, from

$$\sin \theta < \theta < \tan \theta,$$

we get by dividing by $\tan \theta$,

$$\cos \theta < \frac{\theta}{\tan \theta} < 1,$$

and as $\theta \to 0$, $\cos \theta \to 1$ and $\frac{\theta}{\tan \theta}$ always lying between $\cos \theta$ and 1 which come into coincidence, $\frac{\theta}{\tan \theta} = 1$ in the limit, and so $\underset{\theta \to 0}{Lt} \frac{\tan \theta}{\theta} = 1$.

Hence, the results.

2. *Formulæ for* **sin (A + B)** *and* **cos (A + B)** *where A and B are of any magnitude.* (*Generalization of Art. 33*)
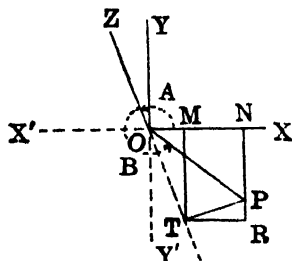
In Article 33, formulæ for sin $(A + B)$ and cos $(A + B)$ were deduced geometrically with a figure in which A and B were acute and $(A + B)$ less than 90°. We now prove them in a more general case.

Let a revolving line, starting from OX, trace out an angle $XOZ = A$ and further trace out an angle $ZOP = B$, so that the total angle traced out is $(A + B)$. From any point P on the final position of the revolving line, PN and PT are drawn perpendiculars to OX and OZ (produced if necessary, as in the above figure), and from T perpendiculars TM and TR are drawn on OX and PN (produced if necessary).

In the figure above, $\angle POT = B - 180°$, and since PN and PT are perpendiculars to OX and OZ respectively, $\angle TPR = \angle TON = 180° - \angle XOZ$, i.e., $180° - A$.

In considering sin $(A + B)$ and cos $(A + B)$ from the triangle NOP, it is to be noted that PN is negative and ON and OP are positive.

If we consider only the positive magnitudes of the sides of the acute-angled triangles OTM, PTR and OPT, then PN with its proper sign may be written as $-(TM - PR)$, and ON with its proper sign may be written as $OM + TR$.

Now, from the figure,

$$\sin (A + B) = \frac{PN}{OP} = -\frac{TM - PR}{OP}$$

I. Tr.—16

$$= -\frac{TM}{OT}\cdot\frac{OT}{CP} + \frac{PR}{PT}\cdot\frac{PT}{OP}$$

$$= -\sin TOM \cos POT + \cos TPR \sin POT$$

$$= -\sin (180° - A) \cos (B - 180°)$$
$$+ \cos (180° - A) \sin (B - 180°)$$

$$= -\sin A (-\cos B) + (-\cos A)(-\sin B)$$

$$= \sin A \cos B + \cos A \sin B.$$

Again,

$$\cos (A + B) = \frac{ON}{OP} = \frac{OM + RT}{OP}$$

$$= \frac{OM}{OT}\cdot\frac{OT}{OP} + \frac{RT}{PT}\cdot\frac{PT}{OP}$$

$$= \cos TOM \cos POT + \sin TPR \sin POT$$

$$= \cos (180° - A) \cos (B - 180°)$$
$$+ \sin (180° - A) \sin (B - 180°)$$

$$= (-\cos A)(-\cos B) + \sin A (-\sin B)$$

$$= \cos A \cos B - \sin A \sin B.$$

3. *Formulæ for* **sin (A − B)** *and* **cos (A − B)** *in a more general case.* (*Generalization of Art. 34*)
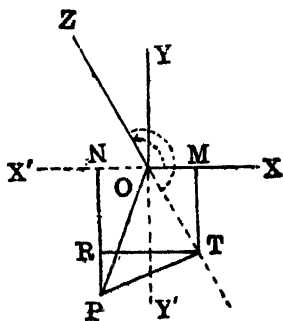
Here, XOZ measured counter-clockwise is A and ZOP measured clockwise has magnitude B so that XOP measured

clockwise is $A - B$. Fron P, PN and PT are drawn perpendiculars on OX and OZ (produced in this figure), and from T, TM and TR are drawn perpendiculars on OX and PN.

In the present figure, magnitudes of the acute angles TOM and POT are $180° - A$ and $B - 180°$ respectively, and noting that $PNOT$ is a cyclic quadrilateral ($\angle^s N$ and T being right angles), $\angle RPT = \angle TOM = 180° - A$ in magnitude.

Now, we see that in considering $\sin(A - B)$ and $\cos(A - B)$ from the triangle NOP, PN and ON are of negative signs.

Hence,

$$\sin(A - B) = \frac{PN}{OP}$$

$$= -\frac{MT + PR}{OP},$$

[where the magnitudes of MT, PR, etc. only are considered]

$$= -\frac{MT}{OT} \cdot \frac{OT}{OP} - \frac{PR}{PT} \cdot \frac{PT}{OP}$$

$$= -\sin TOM \cos POT - \cos RPT \sin POT$$

$$= -\sin(180° - A) \cos(B - 180°)$$
$$\quad - \cos(180° - A) \sin(B - 180°)$$

$$= -\sin A(-\cos B) - (-\cos A)(-\sin B)$$

$$= \sin A \cos B - \cos A \sin B.$$

Similarly,

$$\cos(A - B) = \frac{ON}{OP} \qquad [\text{ where } ON \text{ is taken with proper sign }]$$

$$= -\frac{RT - OM}{OP} \qquad [\text{ where magnitudes only of } RT, OM \text{ etc. are considered }]$$

$$= -\frac{RT}{PT} \cdot \frac{PT}{OP} + \frac{OM}{OT} \cdot \frac{OT}{OP}$$

$$= -\sin RPT \sin POT + \cos TOM \cos POT$$

$$= -\sin(180° - A) \sin(B - 180°)$$
$$\quad + \cos(180° - A) \cos(B - 180°)$$

$$= -\sin A \,(-\sin B) + (-\cos A)(-\cos B)$$
$$= \cos A \cos B + \sin A \sin B.$$

4.   A few particular cases of sin (A ± B), cos (A ± B).

Case I.   In the case A and B are both acute and
$(A + B) > 90°.$

Construction is 'same as in Art. 33.   Here, Q, the foot of the perpendicular will fall on XO produced.

$$\angle TPR = 90° - \angle TRP = \angle TRO = \angle ROS = \angle A.$$

$$\sin (A + B) = \sin XOP$$
$$= \frac{PQ}{OP} = \frac{QT + TP}{OP}$$
$$= \frac{RS + PT}{OP} = \frac{RS}{OP} + \frac{PT}{OP}$$
$$= \frac{RS}{OR} \cdot \frac{OR}{OP} + \frac{PT}{PR} \cdot \frac{PR}{OP}$$
$$= \sin A \cos B + \cos TPR \sin B$$
$$= \sin A \cos B + \cos A \sin B.$$
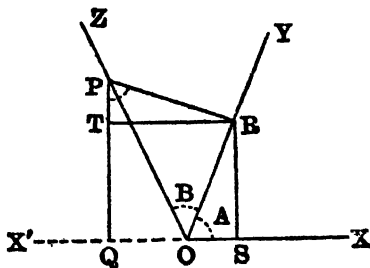
$$\cos (A + B) = \cos XOP = -\frac{OQ}{OP} \qquad \text{[Magnitude of } OQ \text{ being considered]}$$
$$= -\frac{SQ - SO}{OP} = \frac{OS}{OP} - \frac{SQ}{OP} = \frac{OS}{OP} - \frac{TR}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

Case II. In the case A is obtuse and B is acute and
$(A + B) < 180°$.

Construction is same as in Art. 33.

Here, $\angle TPR = 180° - \angle RPQ = \angle ROQ = 180° - A.$

∴ $\sin TPR = \sin A$; $\cos TPR = -\cos A.$

$$\sin (A + B) = \sin XOP = \frac{PQ}{OP} = \frac{QT - PT}{OP} = \frac{RS - PT}{OP}$$

$$= \frac{RS}{OP} - \frac{PT}{OP} = \frac{RS}{OR} \cdot \frac{OR}{OP} - \frac{PT}{PR} \cdot \frac{PR}{OP}$$

$$= \sin A \cos B - \cos TPR \sin B.$$

$$= \sin A \cos B + \cos A \sin B.$$

$\cos (A + B) = \cos XOP = -\frac{OQ}{OP}$ [Magnitude of OQ being
                                                           considered]

$$= -\frac{OS + SQ}{OP} = -\frac{OS}{OP} - \frac{SQ}{OP}$$

$$= -\frac{OS}{OR} \cdot \frac{OR}{OP} - \frac{TR}{PR} \cdot \frac{PR}{OP}$$

$$= \cos A \cos B - \sin TPR \sin B$$

$$= \cos A \cos B - \sin A \sin B.$$

Case III. In the case A and B are both obtuse and
$(A - B)$ *is acute.*

Construction is same as in Art. 34.

Here, $\angle TPR = \angle ROS = 180° - A.$
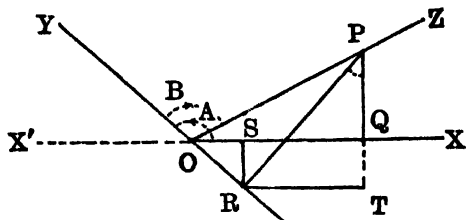
$\sin (A - B) = \sin POQ$

$$= \frac{PQ}{OP} = \frac{PT - RS}{OP}$$

$$= \frac{PT}{OP} - \frac{RS}{OP} = \frac{PT}{PR} \cdot \frac{PR}{OP} - \frac{RS}{OR} \cdot \frac{OR}{OP}$$

$$= \cos TPR \sin POR - \sin ROS \cos POR$$

$$= \cos (180° - A) \sin (180° - B).$$

$$\qquad - \sin (180° - A) \cos (180° - B)$$

$$= - \cos A \sin B - \sin A \, (- \cos B)$$

$$= \sin A \cos B - \cos A \sin B.$$

$\cos (A - B) = \cos POQ$

$$= \frac{OQ}{OP} = \frac{OS + SQ}{OP} = \frac{OS + RT}{OP} = \frac{OS}{OP} + \frac{RT}{OP}$$

$$= \frac{OS}{OR} \cdot \frac{OR}{OP} + \frac{RT}{PR} \cdot \frac{PR}{OT}$$

$$= \cos ROS \cdot \cos POR + \sin TPR \cdot \sin POR$$

$$= \cos (180° - A) \cos (180° - B)$$

$$\qquad + \sin (180° - A) \sin (180° - B)$$

$$= (-\cos A)(-\cos B) + \sin A \sin B$$
$$= \cos A \cos B + \sin A \sin B.$$

Note. Other particular cases of the above four formulæ can easily be proved exactly in the same way by drawing the corresponding figures in each case and making the same construction as in Arts. 33 and 34 for $(A+B)$ and $(A-B)$ respectively.

5. An alternative Method of proof for sin $(A \pm B)$, cos $(A \pm B)$.   [See *Arts. 33, 34*]
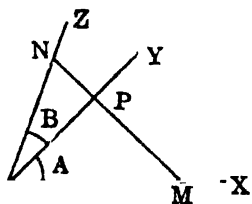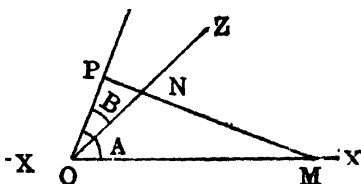
Fig. (i)   Fig. (ii)

Let $\angle XOY = A$; $\angle YOZ = B$; in Fig. (i), $\angle XOZ = A + B$ ($< 90°$); in Fig. (ii), $\angle XOZ = A - B$ ($A > B$) [A and B being positive and acute].

Through any point P on OY, the common arm of two angles, draw a straight line MN perpendicular to OY, meeting OX in M and OZ in N.

Then, $\triangle MON = \triangle MOP \pm \triangle NOP$.

$\therefore$ $\tfrac{1}{2}OM.ON \sin (A \pm B) = \tfrac{1}{2}OM.OP \sin A \pm \tfrac{1}{2}ON.OP \sin B$
$$[\; Art. \; 88(i) \;]$$

$\therefore$ $\sin (A \pm B) = \sin A \cdot \dfrac{OP}{ON} \pm \dfrac{OP}{OM} \sin B$

$$= \sin A \cos B \pm \cos A \sin B.$$

$\cos (A \pm B) = \cos MON = \dfrac{OM^2 + ON^2 - MN^2}{2OM.ON}$ [*Art. 83*]

$$= \dfrac{(OP^2 + PM^2) + (OP^2 + PN^2) - (MP \mp PN)^2}{2OM.ON}$$

$$= \frac{OP^2 \mp MP.PN}{OM.ON} = \frac{OP}{OM}.\frac{OP}{ON} \mp \frac{MP}{OM}.\frac{PT}{ON}$$

$$= \cos A \cos B \mp \sin A \sin B.$$

6. Geometrical proof of the expansion of tan $(A+B)$.

The figure and the construction are the same as in Art. 33.

$$\tan (A+B) = \frac{PQ}{OQ} = \frac{RS+PT}{OS-TR}$$

$$= \frac{\dfrac{RS}{OS} + \dfrac{PT}{OS}}{1 - \dfrac{TR}{OS}} = \frac{\dfrac{RS}{OS} + \dfrac{PT}{OS}}{1 - \dfrac{TR}{TP}.\dfrac{TP}{OS}}.$$

Now, $\quad \dfrac{RS}{OS} = \tan A$ and $\dfrac{TR}{TP} = \tan TPR = \tan A.$

The triangles ROS, TPR are similar.

$$\therefore \quad \frac{TP}{OS} = \frac{PR}{OR} = \tan B.$$

$$\therefore \quad \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Note. Similarly the expansion of tan $(A-B)$ can be proved geometrically from the figure and construction of Art. 34.

7. Geometrical proof of the formulæ for 2A.
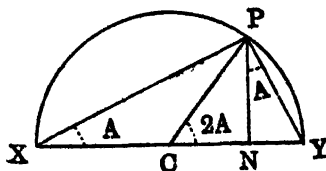
Let XPY be a semi-circle, XY the diameter and C the centre.

Draw PN perpendicular to XY.

Let $\angle PXY = A$; then $\angle PCY = 2A$.

$\angle NPY = 90° - \angle PYN = \angle PXY$.

$\sin 2A = \dfrac{PN}{CP} = \dfrac{2PN}{2CP} = \dfrac{2PN}{XY} = 2\dfrac{PN}{XP}\cdot\dfrac{XP}{XY}$

$\quad = 2\sin PXN.\cos PXY = 2\sin A\cos A.$

$\cos 2A = \dfrac{CN}{PC} = \dfrac{2CN}{2CP} = \dfrac{2CN}{XY} = \dfrac{CN+CN}{XY}$

$\quad = \dfrac{(XN-XC)+(CY-YN)}{XY} = \dfrac{XN-YN}{XY}$

$\quad = \dfrac{XN}{XP}\cdot\dfrac{XP}{XY} - \dfrac{YN}{PY}\cdot\dfrac{PY}{XY}$

$\quad = \cos A.\cos A - \sin A.\sin A$

$\quad = \cos^2 A - \sin^2 A.$

$\tan 2A = \dfrac{PN}{CN} = \dfrac{2PN}{2CN} = \dfrac{2PN}{XN-YN}$

$\dfrac{2\dfrac{PN}{XN}}{1-\dfrac{YN}{XN}} = \dfrac{2\dfrac{PN}{XN}}{1-\dfrac{YN}{PN}\cdot\dfrac{PN}{XN}}$

$\quad = \dfrac{2\tan A}{1-\tan A.\tan A} = \dfrac{2\tan A}{1-\tan^2 A}.$

8. Trigonometrical ratios of generalized angle defined by projections.
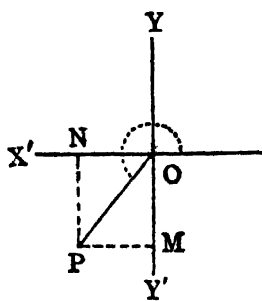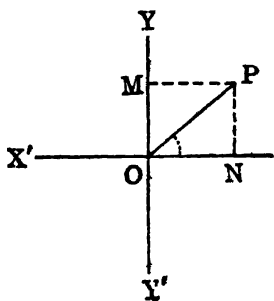
Let XOX' and YOY' be a pair of rectangular axes intersecting at the point O and let an angle θ, of any magnitude (positive or negative) be generated by the revolution of OP from its initial position OX to its present position. Then the trigonometrical ratios of the generalized angle θ are defined as follows :

$$\sin \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{OP}$$

$$\cos \theta = \frac{\text{projection of } OP \text{ on } x\text{-axis}}{OP}$$

$$\tan \theta = \frac{\text{projection of } OP \text{ on } y\text{-axis}}{\text{projection of } OP \text{ on } x\text{-axis}}$$

$$\operatorname{cosec} \theta = \frac{OP}{\text{projection of } OP \text{ on } y\text{-axis}}$$

$$\sec \theta = \frac{OP}{\text{projection of } OP \text{ on } x\text{-axis}}$$

$$\cot \theta = \frac{\text{projection of } OP \text{ on } x\text{-axis}}{\text{projection of } OP \text{ on } y\text{-axis}}$$

In the above definitions, projection means algebraic projection ; that is, we should consider not only the magnitude but also the sign of the projection ; and with the usual convention the projection would be considered *positive* if they are along OX and OY and considered *negative* if they are along OX' and OY'. By convention, *OP is always considered positive*. From these definitions, the signs of the trigonometrical ratios can be easily determined according to the position of OP in one or other of the four quadrants. In the above figures, the position of OP in two quadrants only (1st and 3rd) are shown.

Note 1. From the above definitions, it is clear that if OX be a fixed line, and if l be the length of any straight line OP inclined at an angle θ to OX, then the *projection of OP along OX is $l \cos \theta$* whatever be the magnitude of the angle θ.

Note 2. The *Addition and Subtraction Theorem* for generalized angles can also be proved by the method of projection.

9. Two important trigonometrical relations.
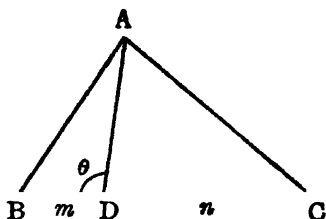
If D be any point in the base BC of a triangle ABC, and if AD divides BC into two parts m and n $(BD = m, CD = n)$ and the angle A into two parts a and β $(\angle BAD = a, \angle CAD = \beta)$ and if the angle ADB be θ, then

(i) $(m+n) \cot \theta = n \cot \beta - m \cot a.$

(ii) $(m+n) \cot \theta = m \cot C - n \cot B.$

We have

$$\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{AD} \cdot \frac{AD}{DC} = \frac{\sin BAD}{\sin ABD} \cdot \frac{\sin ACD}{\sin DAC}$$

$$= \frac{\sin a}{\sin (\theta + a)} \cdot \frac{\sin (\theta - \beta)}{\sin \beta} \left[\begin{array}{l} \because \angle ABD = \pi - (a + \theta). \\ \angle ACD = \theta - \beta. \end{array} \right]$$

$$= \frac{\sin a (\sin \theta \cos \beta - \cos \theta \sin \beta)}{\sin \beta (\sin \theta \cos a + \cos \theta \sin a)}.$$

Dividing the numerator and the denominator by $\sin a \sin \beta \sin \theta$, we have

$$\frac{m}{n} = \frac{\cot \beta - \cot \theta}{\cot a + \cot \theta}.$$

$\therefore \quad (m+n) \cot \theta = n \cot \beta - m \cot a.$

Again,

$$\frac{m}{n} = \frac{\sin BAD}{\sin ABD} \cdot \frac{\sin ACD}{\sin DAC}$$

$$= \frac{\sin (\theta + B)}{\sin B} \cdot \frac{\sin C}{\sin (\theta - C)} \left[\begin{array}{l} \because \angle BAD = \pi - (\theta + B). \\ \angle DAC = \theta - C \end{array} \right]$$

$$= \frac{\sin C (\sin \theta \cos B + \cos \theta \sin B)}{\sin B (\sin \theta \cos C - \cos \theta \sin C)}.$$

Dividing the numerator and the denominator by $\sin B \sin C \sin \theta$, we have

$$\frac{m}{n} = \frac{\cot B + \cot \theta}{\cot C - \cot \theta}.$$

$$\therefore \quad (m + n) \cot \theta = m \cot C - n \cot B.$$

10.   Note on Art. 90.

Let us denote the formulæ of Arts. 82, 83, 84 by (I), (II), (III). We have seen in Art. 90, that (II) can be deduced from (III). We shall now show how any one of these can be deduced from any other of the rest.

To deduce (I) from (III).

Substituting the value of b from the second relation of Art. 84 in the first,

$$a = (c \cos A + a \cos C) \cos C + c \cos B.$$

$$\therefore \quad a (1 - \cos^2 C) = c (\cos A \cos C + \cos B)$$
$$= c \{\cos A \cos C - \cos (A + C)\}$$
$$[\because A + B + C = \pi]$$

$$\therefore \quad a \sin^2 C = c \sin A \sin C. \quad \therefore \quad a/\sin A = c/\sin C.$$

Similarly, substituting the value of c in the first relation, we get

$$a/\sin A = b/\sin B. \quad \text{Hence, etc.}$$

To deduce (II) and (III) from (I).

(i) Putting each of the ratios of Art. 82 equal to k, we get

$$a = k \cdot \sin A \; ; \; b = k \cdot \sin B \; ; \; c = k \cdot \sin C.$$

$$\therefore \quad \frac{b^2 + c^2 - a^2}{2bc} = \frac{k^2 (\sin^2 B + \sin^2 C - \sin^2 A)}{k^2 \cdot 2 \sin B \sin C}$$

$$= \frac{\sin^2 B + \sin (C + A) \sin (C - A)}{2 \sin B \sin C}$$

$$= \frac{\sin B \{\sin B + \sin (C - A)\}}{2 \sin B \sin C}$$

$$[\because \sin (C+A) = \sin (\pi - B) = \sin B \,]$$

$$= \frac{\sin B \{\sin (C + A) + \sin (C - A)\}}{2 \sin B \sin C}$$

$$= \frac{2 \sin B \sin C \cos A}{2 \sin B \sin C} = \cos A.$$

(ii) $b \cos C + c \cos B = k (\sin B \cos C + \sin C \cos B)$

$$= k \sin (B + C) = k \sin A$$

$$= a. \qquad [\because A + B + C = \pi \,]$$

To deduce (I) and (III) from (II).

(i) $\sin^2 A = 1 - \cos^2 A$

$$= 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2}$$

$$= \frac{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}{4b^2 c^2}$$

$$= \frac{K}{4b^2c^2} \text{ say.}$$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{K}{4a^2b^2c^2} \,;$$

similarly, $\frac{\sin^2 B}{b^2}$ and $\frac{\sin^2 C}{c^2}$ each $= \frac{K}{4a^2b^2c^2}.$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2} \,; \text{ hence, etc.}$$

(ii) Adding 2nd and 3rd relations of the formulæ of Art. 83, we get

$$b^2 + c^2 = b^2 + c^2 + 2a^2 - 2ca \cos B - 2ab \cos C.$$

Now, transposing and dividing by $2a$, we get

$$a = b \cos C + c \cos B \,; \text{ etc.}$$

Miscellaneous Examples III

1. The angles of a triangle are as $4 : 5 : 6$. Express them in circular measure.

2. The angles of a triangle are in A.P. and the greatest is double the least ; express the angles in degrees, and in radians.

3. The number of degrees in one of the acute angles of a right-angled triangle is three-tenths of the number of grades in the other ; determine the angles in degrees.

4. Compare the areas of two circles in which the circumference of one is equal to an arc of $60°$ of the other.

5. A railway train is travelling on a curve of half-a-mile radius at the rate of 20 miles an hour ; through what angle has it turned in 10 seconds ?

6. An arc of a circle whose radius is 7 inches, subtends an angle of $15° \, 39' \, 7''$; what angle will an arc of the same length subtend in a circle whose radius is 2 inches ?

Prove the following identities (*Ex. 7 to 22*) :—

7. $\sin^2\theta \tan\theta + \cos^2\theta \cot\theta + 2 \sin\theta \cos\theta = \tan\theta + \cos\theta$.

8. $\sin^2\theta \, (1 + \cot^2\theta) + \cos^2\theta \, (1 + \tan^2\theta) = 2$.

9. $(\tan\theta + \sec\theta)^2 = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$ [*C. U. 1934*]

10. $2 \, (\sin^6\theta + \cos^6\theta) - 3 \, (\sin^4\theta + \cos^4\theta) + 1 = 0$.

11. $\dfrac{\tan x - \cot y}{\tan y - \cot x} = \tan x \cot y$.

12. $(\sin x \cos y + \cos x \sin y)^2$
$\qquad\qquad + (\cos x \cos y - \sin x \sin y)^2 = 1$.

13. $\sin^4 x + \cos^4 x = 1 - 2 \sin^2 x \cos^2 x$.

14. $\sin^8 x - \cos^8 x = (\sin^2 x - \cos^2 x)(1 - 2 \sin^2 x \cos^2 x)$.

15. $\sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta$.

16. $(1 + \sec \theta + \tan \theta)(1 - \sec \theta + \tan \theta) = 2 \tan \theta$.

17. $\dfrac{\cos \theta}{1 - \tan \theta} + \dfrac{\sin \theta}{1 - \cot \theta} = \sin \theta + \cos \theta$.

18. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$.

19. $\cot^2 x \cdot \dfrac{\sec x - 1}{1 + \sin x} + \sec^2 x \cdot \dfrac{\sin x - 1}{1 + \sec x} = 0$.

20. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \operatorname{cosec} x$.

21. $(\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2 = (1 + \sec \theta \operatorname{cosec} \theta)^2$.

22. $\dfrac{1 - \sin \theta \cos \theta}{\cos \theta (\sec \theta - \operatorname{cosec} \theta)} \cdot \dfrac{\sin^2 \theta - \cos^2 \theta}{\sin^3 \theta + \cos^3 \theta} = \sin \theta$.

23. If $a \cos^2 x + b \sin^2 x = c$, show that $\tan x = \pm \sqrt{\dfrac{c - a}{b - c}}$.

24. If $\operatorname{cosec} A + \operatorname{cosec} B + \operatorname{cosec} C = 0$, show that
$(\Sigma \sin A)^2 = \Sigma \sin^2 A$.

25. If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$,
show that $x^2 + y^2 = a^2 + b^2$.

26. Express $\dfrac{\sin x}{\cos^3 x} + \dfrac{\cos x}{\sin^3 x}$ in terms of t,
where t stands for $\tan x$.

27. If $\sin A = \frac{1}{2}$ and $\tan B = \sqrt{3}$, find the value of
$\sin A \cos B + \cos A \sin B$.

28. If $\cos \theta = \frac{2}{3}$, find the value of $\dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$.

29. If $5 \tan \theta = 4$, find the value of
$\dfrac{5 \sin \theta - 3 \cos \theta}{\sin \theta + 2 \cos \theta}$.

30. If $\dfrac{\sin x}{\sin y} = \sqrt{2}, \dfrac{\tan x}{\tan y} = \sqrt{3}$,

find x and y (given that x and y are acute angles).

31. Which of the statements is possible and which impossible, x, y and z being unequal real quantities ?

(i) $\operatorname{cosec} \theta = \dfrac{x^2 + y^2}{2xy}$.   (ii) $\sec \theta = \dfrac{x^2 - y^2}{x^2 + y^2}$.

(iii) $\sin \theta = \dfrac{x^2 + y^2 + z^2}{yz + zx + xy}$. (iv) $\tan \theta = \dfrac{2xy}{x^2 + y^2}$.

32. Eliminate θ from the equations :

(i) $\sin \theta + \cos \theta = a$, $\sin \theta - \cos \theta = b$.

(ii) $\dfrac{x}{a} \cos \theta + \dfrac{y}{b} \sin \theta = 1, \dfrac{x}{a} \sin \theta - \dfrac{y}{b} \cos \theta = 1$.

(iii) $x = a \cos^3 \theta$, $y = b \sin^3 \theta$.

33. If $k \tan \theta = \tan k\theta$, prove that

$$\dfrac{\sin^2 k\theta}{\sin^2 \theta} = \dfrac{k^2}{1 + (k^2 - 1) \sin^2 \theta}.$$

34. If $\sec x \sec y + \tan x \tan y = \sec z$,

then, $\sec x \tan y + \tan x \sec y = \pm \tan z$.

35. Show that $\left(\dfrac{1 + \cot 60°}{1 - \cot 60°}\right)^2 = \dfrac{1 + \cos 30°}{1 - \cos 30°}$.

36. If $\tan x = \dfrac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}$, prove that

$$\sin x = \dfrac{1}{\sqrt{2}} (\sin \theta - \cos \theta).$$

37. Show that the product of $\sin x (1 + \sin x) + \cos x (1 + \cos x)$ and $\sin x (1 - \sin x) + \cos x (1 - \cos x)$ is equal to $2 \sin x \cos x$.

38. Find the height of a chimney when it is found that on walking towards it 250 feet, in a horizontal line through its base, the angular elevation changes from 45° to 75°.

39. The length of a kite string is 250 yards, and the angle of elevation of the kite is 30°. Find the height of the kite.

40. The angle of elevation of the top of a temple at a distance 300 feet is 30° ; find its height.

41. Find the angle of elevation of the sun when the shadow of a pole 60 feet high, is $20 \sqrt{3}$ yards long.

42. The angles of elevation of a tower at two places due east of it and 200 feet apart are 45° and 30° ; find the height of the tower.

43. An aeroplane leaves the earth at the point X and rises at a uniform speed. At the end of 15 seconds, an observer stationed at a distance of 660 feet from X, find the angular elevation of the aeroplane to be 45° ; at what rate in miles per hour is the aeroplane rising ?

44. A ladder 45 feet long just reaches the top of a wall. The ladder makes an angle of 60° with the wall. Find the height of the wall and the distance of the foot of the ladder from the wall.

45. If $\cos A = \frac{4}{5}$, $\cos B = \frac{3}{5}$, find the values of
$$\sin (A + B) \text{ and } \cos (A - B).$$

46. If $\tan A = \frac{5}{12}$ and $\tan B = \frac{9}{40}$, find the values of
$$\sin (A - B) \text{ and } \cos (A - B).$$

47. If $\tan A = \frac{m + n}{m - n}$, and $\tan B = \frac{m - n}{m + n}$, find $\tan (A - B)$.

48. If $\tan (x + y) = \frac{4}{3}$ and $\tan x = \frac{5}{8}$, find $\tan y$.

49. If $\cos \theta = \frac{3}{5}$, find $\sin 2\theta$, $\tan 2\theta$, $\cos \frac{\theta}{2}$.

50. If $\cos x = \frac{1}{4}$, $\cos y = \frac{3}{4}$ (x and y being positive acute angles), find the value of $\cos \frac{1}{2} (x - y)$.

I. Tr.—17

51. If $\sin A = \dfrac{1}{\sqrt{2}}$, $\sin B = \dfrac{1}{\sqrt{3}}$, find the value of

$$\tan \tfrac{1}{2}(A+B) \cot \tfrac{1}{2}(A-B).$$

52. If $\sec x = \tfrac{17}{8}$, $\operatorname{cosec} y = \tfrac{5}{4}$, find $\sec (x+y)$.

53. Prove that

$$\frac{2 \cos 8\theta + 1}{2 \cos \theta + 1} = (2 \cos \theta - 1)(2 \cos 2\theta - 1)(2 \cos 4\theta - 1).$$

54. Show that $a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos (\theta - a)$, if $\tan a = b/a$.

55. If $\sin^4 x + \cos^4 x = 1$, prove that x is zero or a multiple of $\tfrac{1}{2}\pi$.

56. If $\sqrt{2} \cos A = \cos B + \cos^3 B$, and $\sqrt{2} \sin A = \sin B - \sin^3 B$, then $\sin (A - B) = \pm \tfrac{1}{3}$.

57. Prove that $\cos^2 (a - \beta) - \sin^2 (a + \beta) = \cos 2a \cos 2\beta$.

58. Show that $\sin 18° + \cos 18° = \sqrt{2} \cos 27°$.

59. Show that whatever be the value of θ,

$\sin^2(\theta + a) + \sin^2(\theta + \beta) - 2 \cos (a - \beta) \sin (\theta + a) \sin (\theta + \beta)$ is independent of θ.

60. Show that

(i) $\dfrac{\sin a}{\sin (a - \beta) \sin (a - \gamma)} + \dfrac{\sin \beta}{\sin (\beta - \gamma) \sin (\beta - a)}$

$$+ \dfrac{\sin \gamma}{\sin (\gamma - a) \sin (\gamma - \beta)} = 0.$$

(ii) $\tan (\beta + \gamma - 2a) + \tan (\gamma + a - 2\beta) + \tan (a + \beta - 2\gamma)$

$$= \tan (\beta + \gamma - 2a) \tan (\gamma + a - 2\beta) \tan (a + \beta - 2\gamma).$$

61. If $\tan \frac{1}{2}\theta = \tan^3 \frac{1}{2}\phi$ and $\tan \phi = 2 \tan a$, show that

$$\theta + \phi = 2a.$$

62. (i) If $\tan^2 x = 2 \tan^2 y + 1$, then $\cos 2x + \sin^2 y = 0$.

(ii) If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$,
prove that $\sin A = \sin B = \sin C$.

63. Show that $\tan 20° \tan 40° \tan 80° = \tan 60°$.

64. If $a + \beta + \gamma = 0$, prove that

$$\cos a + \cos \beta + \cos \gamma = 4 \cos \frac{a}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 1.$$

65. If in any triangle, $\tan \phi = \dfrac{a-b}{a+b} \cot \frac{1}{2}C$, prove that

$$c = (a+b) \sin \frac{1}{2}C \sec \phi.$$

66. If $\cos \theta = \dfrac{a \cos \phi - b}{a - b \cos \phi}$, then $\dfrac{\tan \dfrac{\theta}{2}}{\sqrt{a+b}} = \dfrac{\tan \dfrac{\phi}{2}}{\sqrt{a-b}}.$

67. If $a + \beta + \gamma = \frac{1}{2}\pi$, prove that

$$\sin^2 a + \sin^2 \beta + \sin^2 \gamma + 2 \sin a \sin \beta \sin \gamma = 1.$$

68. If $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$,
show that one of the quantities $A \pm B \pm C$ is an odd multiple
of π.

69. Show that $\sec x = \dfrac{2}{\sqrt{2 + \sqrt{2 + 2 \cos 4x}}}.$

70. If $a \sin (\theta + a) = b \sin (\theta + \beta)$, prove that

$$\cot \theta = \frac{a \cos a - b \cos \beta}{b \sin \beta - a \sin a}.$$

71. If $\tan \beta = \dfrac{n \sin a \cos a}{1 - n \sin^2 a}$, show that

$$\tan (a - \beta) = (1 - n) \tan a.$$

In any triangle, prove that (*Ex. 72 to 77*) :

72. $\dfrac{\cos A}{c \cos B + b \cos C} + \dfrac{\cos B}{a \cos C + c \cos A}$

$$+ \dfrac{\cos C}{b \cos A + a \cos B} = \dfrac{a^2 + b^2 + c^2}{2abc}.$$

73. $\dfrac{\tan \dfrac{A}{2}}{(a-b)(a-c)} + \dfrac{\tan \dfrac{B}{2}}{(b-c)(b-a)} + \dfrac{\tan \dfrac{C}{2}}{(c-a)(c-b)} = \dfrac{1}{\triangle}.$

74. $\sin 3A \sin (B - C) + \sin 3B \sin (C - A)$
$$+ \sin 3C \sin (A - B) = 0.$$

75. $\cot B + \dfrac{\cos C}{\sin B \cos A} = \cot C + \dfrac{\cos B}{\sin C \cos A}.$

76. $c = (a - b) \sec \theta$, where $\tan \theta = \dfrac{2 \sqrt{ab}}{a - b} \sin \dfrac{C}{2}$.

77. $a (\cos B \cos C + \cos A) = b (\cos C \cos A + \cos B)$
$$= c (\cos A \cos B + \cos C).$$

78. If in a triangle, $c (a + b) \cos \dfrac{B}{2} = b(a + c) \cos \dfrac{C}{2}$, show that the triangle is isosceles.

79. If in a triangle, a, b, c be in A.P. and the greatest angle exceeds the least by 90°, prove that
$$a : b : c = \sqrt{7} - 1 : \sqrt{7} : \sqrt{7} + 1.$$

80. In the solution of triangles there can be no ambiguity except when an angle is determined by the sine or cosecant, and in no case whatever, when the triangle has one right angle ; prove this.   [*Cambridge*]

81. If $\sin (\pi \cos \theta) = \cos (\pi \sin \theta)$, prove that
$$\cos \left(\theta \pm \dfrac{\pi}{4}\right) = \dfrac{1}{2 \sqrt{2}}.$$

82. If $\sin (\pi \cot \theta) = \cos (\pi \tan \theta)$, prove that either $\operatorname{cosec} 2\theta$ or $\cot 2\theta$ is equal to $n + \frac{1}{4}$, n being an integer.

83. If a and β be the different values of θ which satisfy the equation $a \cos \theta + b \sin \theta = c$, prove that

$$\sin (a + \beta) = \frac{2ab}{a^2 + b^2}.$$

84. Find all the values of θ which satisfy the equation

$$\sin \theta + \sin 2\theta + \sin 3\theta = 1 + \cos \theta + \cos 2\theta.$$

85. Prove that in any triangle,

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}.$$

86. If $r : R : r_1 = 2 : 5 : 12$, show that the triangle is right-angled.

87. If the angle of elevation of a cloud observed from a point at a height h above the surface of a lake be ϕ and the angle of depression of its image in the lake be θ, prove that the height of the cloud above the lake is $h \dfrac{\sin (\theta + \phi)}{\sin (\theta - \phi)}$, assuming that the image is vertically as much below the surface as the cloud is above it.

[*B. H. U. I. 1931* ; *A. U. 1942*]

88. The elevation of a tower due north of a station at A is a and at a station B due west of A is β. Prove that its altitude is $\dfrac{AB \sin a \sin \beta}{\sqrt{\sin^2 a - \sin^2 \beta}}$. [*B. H. U. I. 1934*]

89. A man walks along a straight road and observes that the greatest angle subtended by two objects is a ; from the point where this greatest angle is subtended, he walks a distance c along the road and finds that the two objects are now in a straight line which makes an angle β with the road. Prove that the distance between the objects is

$c \sin a \sin \beta \sec \dfrac{a + \beta}{2} \sec \dfrac{a - \beta}{2}$. [*B. H. U. I. 1936*]

90. On the bank of a river is a column 200 ft. high supporting a statue 30 ft. high. To an observer on the opposite bank with his eye on the level of the ground the statue subtends an angle equal to that subtended by a man 6 ft. high standing at the base of the column ; determine the breadth of the river. [*B. H. U. I. 1941*]

91. If O be a point inside the triangle ABC such that the angles AOB, BOC, COA are each equal to $120°$ and if $OA = x$, $OB = y$ and $OC = z$, show that

$$a^2(y - z) + b^2(z - x) + c^2(x - y) = 0.$$

92. If $\tan \theta = \dfrac{\tan x + \tan y}{1 + \tan x . \tan y}$, prove that

$$\sin 2\theta = \frac{\sin 2x + \sin 2y}{1 + \sin 2x \sin 2y}.$$

93. Show that

$$\frac{\sin \theta - \sin 3\theta}{\cos \theta + \cos 3\theta} + \frac{\sin \theta + \sin 3\theta}{\cos \theta - \cos 3\theta} = 2 \cot 2\theta.$$

94. Prove that in the triangle ABC

$$\frac{a^2(b^2 + c^2 - a^2)}{\sin 2A} = \frac{b^2(c^2 + a^2 - b^2)}{\sin 2B} = \frac{c^2(a^2 + b^2 - c^2)}{\sin 2C}.$$

95. Show that

$$\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}.$$

96. If $x + y = 3 - \cos 4\theta$, and $x - y = 4 \sin 2\theta$,

show that $\sqrt{x} + \sqrt{y} = 2$.

97. Show that

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$$

where $A = 2\beta + \gamma - 3\alpha$, $B = 2\gamma + \alpha - 3\beta$, $C = 2\alpha + \beta - 3\gamma$.

98. Prove that in the triangle ABC,
$$BC^2 \cos 2B + CA^2 \cos 2A + 2BC.CA \cos (A - B) = AB^2.$$

ANSWERS

Examples I. [Pages 11-14]

1. (i) first quadrant. (ii) third quadrant.

   (iii) second quadrant. (iv) fourth quadrant.

2. (i) $61^g 34^l 44^{lll}4$. (ii) $175° 49' 1''·776$.

3. (i) ·253775π. (ii) $\frac{33}{370}\pi$.

4. $82° 30'$; $91^g 66^l 6^{lll}·6$; $\frac{11}{24}\pi$.   5. $a : \beta = 5\pi : 24$.

6. $\frac{1}{2}\left(1-\frac{\pi}{180}\right)$.   7. $6°$ and $9°$.   8. $\frac{1}{90}\left(D+\frac{M}{60}\right)-\frac{1}{100}\left(G+\frac{m}{100}\right)$.

9. $\frac{567}{1100}$ nearly.   10. $20°$ and $30°$.   12. $20^g, 40^g, 80^g$.

13. $27°, 9°, 18°$.   14. (i) At $28\frac{4}{11}$ min. and 48 min. past 7.

   (ii) At 7-10.   15. $20°, 60°, 100°$.   16. $\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}$; $\frac{\pi}{21}, \frac{4\pi}{21}, \frac{16\pi}{21}$.

17. $45°, 60°, 120°, 135°$.   18. 9.

19. mx and nx where $x=\dfrac{2(10pm-9qn)}{mn(10p-9q)}$.   20. $\frac{3}{4}$.

21. 3 and 6.   22. $51·41$ miles per hour (nearly).

23. 66444 miles per hour (nearly) ; 431445 miles (nearly).

24. $76·8$ ft. (nearly).   25. 3959 miles (nearly).   26. 33 ft.

27. 360 yds.

Examples II. [Pages 24-26]

25. $(\sin \theta - \cos \theta)^2$.   26. $\dfrac{1}{\tan^4\theta} - \tan^4\theta$.   31. $\dfrac{a^2-b^2}{a^2+b^2}$.

33. $\pm\dfrac{\sqrt{\sec^2 a-1}}{\sec a}$; $\pm\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$.   34. $1\frac{1}{3}$.   36. $\frac{1}{2}$.   37. 1 or $\frac{1}{2}$.

39. $\dfrac{a^2-b^2}{2ab}$; $\dfrac{a^2+b^2}{a^2-b^2}$.   43. (i) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.   (ii) $xy=c^2$.

   (iii) $(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2$.

   (iv) $(a'b-b'c)(ab'-bc')=(aa'-cc')^2$.

Examples III. [Pages 35-36]

7. $\dfrac{\sqrt{3}}{2}$.　　8. (i) 60°.　　　(ii) 45°.　　　(iii) 30° (There is another

angle which is not one of the standard angles).

(iv) 45°.　　(v) 30°.　　(vi) 30°.　　(vii) 30°.

9. $\theta = 52\frac{1}{2}°$, $\phi = 7\frac{1}{2}°$.　　　　　　10. $a = 50°$, $\beta = 10°$.

11. $A = 22\frac{1}{2}°$, $B = 67\frac{1}{2}°$, $C = 45°$.　　　12. (i) $-\frac{14}{3}$.　　(ii) 1.

Examples IV. [Pages 49-51]

1. $\dfrac{1}{2}$; $-\dfrac{1}{\sqrt{3}}$; $\dfrac{2}{\sqrt{3}}$; -1.　2. $-\dfrac{1}{\sqrt{2}}$; $-\dfrac{2}{\sqrt{3}}$; $-\dfrac{1}{\sqrt{3}}$; $\dfrac{\sqrt{3}}{2}$.　　3. 0.

4. $\dfrac{\sqrt{3}}{2}$.　5. (i) 1.　　(ii) ± 2, $\pm\dfrac{2}{\sqrt{3}}$.　10. $\tan^2\theta$; 1.　　12. (i) 2.

(ii) 1.　　(iii) $\sin x$ or 0 according as n is odd or even.　13. $\frac{51}{56}$.

14. $\dfrac{\sqrt{40}}{3}$.　15. (i) $\cot 26°$.　(ii) $\cos 25°$.　(iii) $\csc 39°$.　(iv) $\cos\dfrac{\pi}{9}$.

16. (i) 300°.　　(ii) 480°.　　　　17. (i) 60°.　(ii) 120°, 240°.

(iii) 30°, 150°, 210°, 330°.　(iv) 30°, 150°.　(v) 30°, 135°, 150°, 315°.

Examples V. [Pages 56-59]

1. $100\sqrt{3}$ ft.　　2. 2·89... miles ; $2\frac{1}{2}$ miles.　　3. $20\sqrt{3}$ ft. ; 120 ft.

4. $20\sqrt{3}$ ft. ; 20 ft.　　5. $30\sqrt{2}$ ft.　　6. $400(\sqrt{3}+1)$ yds.

7. $40\sqrt{3}$ ft.　　　8. $\frac{1}{2}(3+\sqrt{3})$ miles.　9. 22·3 miles nearly.

10. 94·64 ft. nearly.　　11. 47·32 ft. nearly.　12. 60 miles per hour.

13. $50\sqrt{6}$ ft.　　　　14. $40\sqrt{6}$ ft. ; $40\sqrt{2}(\sqrt{7}+1)$ ft.

15. $\frac{1}{2}(\sqrt{3}+1)$ miles.　　16. $5\sqrt{13}$ miles.

17. 241·6... ft. ; 91·6... ft.　18. 5·25... miles per hour.

19. 367·38 ft.　　　　20. $\frac{1}{4}\sqrt{6}(\sqrt{5}+1)$.

22. 2 miles.　　　　23. 13·66 ft.

Examples VI. [Pages 68-70]

21. $\sin A \cos B \cos C - \sin B \cos C \cos A + \sin C \cos A \cos B$

$$+ \sin A \sin B \sin C \; ;$$

$$\dfrac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B + \tan C \tan A - \tan B \tan C}.$$

22. $\dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}$.

Examples VIII. [Pages 79-81]

27. (i) a.

Examples IX. [Pages 86-87]

16. $\dfrac{b^2-a^2}{b^2+a^2}$. 17. (i) $2\sin\frac{1}{2}A\sqrt{1+\sin A}+\sqrt{1-\sin A}$.

(ii) No ; $2\sin\frac{1}{2}\theta=\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}$.

Examples XI. [Pages 110-111]

1. $n\pi\pm\dfrac{\pi}{4}$, *i.e.*, $(2k+1)\dfrac{\pi}{4}$. 2. (i) $n\pi\pm\dfrac{\pi}{4}$. (ii) $n\pi\pm\dfrac{\pi}{3}$.

3. $2n\pi\pm\dfrac{\pi}{3}$, $(2k+1)\pi$. 4. $\dfrac{n\pi}{2}+(-1)^n\dfrac{\pi}{12}$.

5. $n\pi+\dfrac{\pi}{4}$, or, $n\pi+(-1)^n\dfrac{\pi}{6}$. 6. $\dfrac{n\pi}{3}$, or, $n\pi\pm\dfrac{\pi}{6}$.

7. $\dfrac{r\pi}{m+(-1)^r n}$. 8. $(2n+1)\dfrac{\pi}{2}$, or, $(2n+1)\dfrac{\pi}{4}$, or, $(2n+1)\dfrac{\pi}{8}$.

9. $n\pi-\dfrac{\pi}{4}$, or, $\dfrac{n\pi}{2}+(-1)^n\dfrac{a}{2}$, where $\sin a=\dfrac{\sqrt{5}-1}{2}$. 10. $\frac{1}{4}\pi$.

11. $n\pi+\dfrac{\pi}{4}$. 12. $(4n+1)\dfrac{\pi}{8}$. 13. $2n\pi+\dfrac{5\pi}{12}$, or, $2n\pi-\dfrac{\pi}{12}$.

14. $(2n+1)\dfrac{\pi}{4}$, or, $n\pi\pm\dfrac{\pi}{6}$. 15. $2n\pi+\dfrac{\pi}{2}$, or, $2n\pi-\beta$, where β is a positive acute angle whose sine is $\frac{2}{5}$. 16. $kn\pi$. 17. $n\pi\pm\frac{1}{6}\pi$.

18. $(4n+1)\dfrac{\pi}{12}$, $(n\neq 3m+2)$. 19. $2n\pi+\frac{7}{12}\pi$, or, $2n\pi+\frac{1}{12}\pi$.

20. $-\frac{2}{3}\pi$, $-\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{1}{2}\pi$. 21. $\frac{1}{2}(n\pi+a)$, where $\tan a=2$.

22. $2n\pi$. 23. $2n\pi$, $\frac{1}{5}(4n+1)\pi$. 24. $90°$, $450°$, $810°$.

25. $\frac{1}{4}\pi$, $\frac{1}{2}\pi$. 27. (i) $\frac{1}{2}n\pi+\frac{1}{4}\pi$; $2n\pi\pm\frac{2}{3}\pi$.

(ii) 0, $\pm\dfrac{\pi}{12}$, $\pm\dfrac{\pi}{6}$, $\pm\dfrac{\pi}{4}$. (iii) $\dfrac{n\pi}{3}$; $n\pi\pm\tan^{-1}\dfrac{1}{\sqrt{2}}$. (iv) $2n\pi-a$, $\dfrac{4n-1}{2}\pi+a$.

(v) $2n\pi$, or, $2n\pi-\frac{1}{2}\pi$. (vi) $(2n+1)\dfrac{\pi}{2}$, $\dfrac{4n+1}{14}\pi$, $\dfrac{4n-1}{6}\pi$.

(vii) $n\pi+\dfrac{a}{2}$; $(2n+1)\dfrac{\pi}{6}-\dfrac{a}{6}$. 28. $n\pi+(-1)^n 21° 48'-68° 12'$.

29. (i) $a=\beta=\frac{1}{2}\pi$; or, $a=\frac{2}{3}\pi$, $\beta=-\frac{1}{3}\pi$.

(ii) $a=\frac{1}{3}\pi$, $\beta=\frac{1}{12}\pi$; or, $a=1\frac{1}{12}\pi$, $\beta=\frac{2}{3}\pi$;

or, $a=\frac{2}{3}\pi$, $\beta=\frac{5}{12}\pi$; or, $a=\frac{7}{12}\pi$, $\beta=-\frac{1}{3}\pi$.

Examples XII. [Pages 119-121]

22. (i) 1. (ii) 0. (iii) $\dfrac{x+y}{1-xy}$. 23. $y=\dfrac{4x(1-x^2)}{1-6x^2+x^4}$.

24. $(x-y)(1+yz)=(y-z)(1+xy)$. 25. (i) $\frac{1}{2}$, or, -8. (ii) $\dfrac{a-b}{1+ab}$.

(iii) $\pm \dfrac{\sqrt{5}}{3}$.   (iv) $\pm \dfrac{1}{\sqrt{3}}$.   (v) $\frac{4}{3}$, or, $-\frac{8}{3}$.   (vi) $\pm \frac{1}{14}\sqrt{21}$

(vii) 0, or, $\frac{1}{2}$.   (viii) 0, $\pm\frac{1}{2}$.   (ix) $2 - \sqrt{3}$.   (x) $\dfrac{6 + \sqrt{6}}{3}$.

Miscellaneous Examples I. [Pages 122-123]

2. $\pm\sqrt{\dfrac{b^2 - c^2}{a^2 - c^2}}$.   19. $a^2 + b^2 = 2(1 + c)$.

Examples XIII(a). [Pages 135-137]

1. (i) 6. (ii) -3.   2. -2.   5. $\dfrac{n}{n-1}$.   9. (i) 1. (ii) $1\frac{1}{2}$.

10. $\overline{1}\cdot 1173942,\ \cdot 3861209$.   13. $2\cdot 425805$.   14. $\cdot 41369$.

15. (i) $\overline{1}\cdot 8969092$. (ii) $\cdot 898665$.   16. $39\cdot 879$.

17. (i) 18. (ii) 6. (iii) 25.   18. (i) 24. (ii) 4. (iii) 79.

19. (i) $\dfrac{\log 2}{\log 3}$, *i.e.*, $\cdot 63......$ (ii) $4 + \dfrac{\log 7}{\log 8}$, *i.e.*, $5\cdot 77...$

(iii) $\dfrac{2\log 7 - 3\log 3}{6\log 5 - \log 7 - 2\log 3}$, *i.e.*, $\cdot 108......$

(iv) $x = \dfrac{\log 3}{\log 3 - \log 2} = 2\cdot 71$ nearly, $y = \dfrac{\log 2}{\log 3 - \log 2} = 1\cdot 71$ nearly.

(v) $\dfrac{2b(2a - b)}{5ab + 3ac - 2b^2 - bc}$ and $\dfrac{2ab}{5ab + 3ac - 2b^2 - bc}$,

where $a = \log 2,\ b = \log 3,\ c = \log 7$.

20. (i) $\log x = \dfrac{a + 3b}{5}$; $\log y = \dfrac{a - 2b}{5}$.

Examples XIII(b). [Pages 142-144]

1. $3\cdot 2766077$.   2. $1\cdot 3686646$.   3. $37\cdot 6018$.   4. $\cdot 7400827$.

5. $\cdot 8455104$; $82°\ 17'\ 21''$.   6. $\cdot 7928863$.

7. $9\cdot 8440554,\ 10\cdot 1559446$.   8. $36°\ 24'\ 36''$.

9. $58°\ 13'\ 55''$.   10. $9\cdot 6198509$; $22°\ 36'\ 28''$.

12. $10\cdot 0957589$.   13. $9\cdot 9147384$.   14. $9\cdot 8718486$.

16. $\theta = 50°\ 7'\ 48''$ nearly.   17. $\cdot 2394$.

Examples XIV(a). [Pages 157-160]

23. $120°$.   24. $A = 60°$.   29. $A = 90°,\ B = 30°,\ C = 60°$.

39. $\sqrt{\dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}}$.   40. 84.

Examples XIV(b). [Pages 166-168]

15. $r = 4$; $R = 8\frac{1}{8}$.

Examples XV(a). [Pages 172-173]

1. $35°\ 5'\ 49''$.   2. $102°\ 1'\ 26''$.   3. $58°\ 59'\ 33''$.

4. $104°\ 30'$; $46°\ 36'$; $28°\ 54'$.   5. (i) $88°\ 59'\ 40'9''$.

(ii) $78°\ 27'\ 46'86''$.   6. (i) $48°\ 11'\ 23''$; $58°\ 24'\ 43''$; $73°\ 23'\ 54''$.

(ii) $132°\ 34'\ 24''$.   7. $A=120°$, $B=45°$, $C=15°$.

8. $A=45°$, $B=30°$, $C=105°$.   9. $A=60°$, $B=38°\ 11'$, $C=81°\ 49'$.

10. $A=105°$, $B=45°$, $C=30°$.   11. $(\sqrt{3}+1):\sqrt{6}:(\sqrt{3}-1)$.

13. $\sqrt{5}+1:\sqrt{5}-1$.   14. $3:4:5$.

Examples XV(b). [Pages 176-178]

1. $B=38°\ 12'\ 48''$, $C=21°\ 47'\ 12''$.

2. $B=56°\ 19'\ 46'3''$, $C=63°\ 40'\ 13'7''$.

3. $A=117°\ 38'\ 45''$, $B=27°\ 38'\ 45''$.

4. $A=94°\ 42'\ 54''$, $B=25°\ 17'\ 6''$.

5. $B=71°\ 44'\ 29'5''$, $C=48°\ 15'\ 30'5''$.

6. (i) $70°\ 53'\ 36''$; $49°\ 6'\ 14''$.   (ii) $74°\ 13'\ 50''$, $35°\ 16'\ 10''$.

(iii) $A=64°\ 21'$, $B=77°\ 25'$, $c=27'39$.

7. (i) $B=78°\ 17'\ 39'6''$, $C=49°\ 36'\ 20'4''$.

(ii) $116°\ 33'\ 54''$; $26°\ 33'\ 54''$.

8. $A=B=75°$, $C=30°$, $b=2\sqrt{6}$.   9. $\sqrt{6}$, $15°$, $105°$.

10. (i) $A=45°$, $B=75°$, $c=\sqrt{6}$.   (ii) $A=80°$, $B=90°$.

11. $27'0375$.   12. $172'6436$ ft.   13. $79'063$.

14. (i) $A=31°\ 20'$, $b=185$, $c=192$.

(ii) $b=18'46$, $c=37'16$, $C=70°\ 30'$.   (iii) $b=118'9$, $c=117'2$.

15. $C=75°$, $a=c=2\sqrt{3}+2$.   16. $C=105°$, $a=\sqrt{2}$, $c=\sqrt{3}+1$.

17. $72°$, $72°$, $36°$; each side $=\sqrt{5}+1$.   18. 8, 1.

Examples XV(c). [Pages 184-185]

1. (i) One solution.   (ii) Ambiguous; two solutions.

(iii) No solution.   (iv) One solution (right-angled triangle).

2. (i) $C=75°$, $A=60°$, $a=\sqrt{6}$ $\Big\}$ (ii) $60°$, or, $120°$.
or $C=105°$, $A=30°$, $a=\sqrt{2}$ $\Big\}$

3. $A=45°$, $C=75°$, $c=\sqrt{3}+1$. (no ambiguity).   4. Impossible.

8. $C=58°\ 56'\ 56'3''$ $\Big\}$ or, $C=121°\ 3'\ 3'7''$ $\Big\}$
$A=87°\ 48'\ 3'7''$ $\Big\}$ $A=25°\ 41'\ 56'3''$ $\Big\}$

9. $B=34°\ 27'$, $C=100°\ 33'$.

10. $A=5°\ 44'\ 21''$.   11. $A=33°\ 39'\ 34''$, $B=86°\ 20'\ 26''$.

12. $A=80°\ 36'$, $C=64°\ 14'$; or, $A=29°\ 4'$, $C=115°\ 46'$.

Miscellaneous Examples II.   [Pages 186-188]

11. 4, 5, 6.

14. $B = 44° 25' 39''$, or, $135° 34' 21''$.

21. $\frac{1}{3}\{n\pi + \frac{1}{2}\pi - (a+b+c)\}$.

24. $\frac{1}{3}(n\pi + \frac{1}{4}\pi)$.

Examples XVI.   [Pages 213-214]

4. $\theta = \frac{1}{4}\pi$.

5. $x = 38° 10'$ nearly.

6. $\frac{1}{4}\pi$.

7. $-\cdot 37$ nearly.

8. (i) $x = 0$.   (ii) $46° 25'$ (nearly) and $90°$.   (iii) $22\frac{1}{2}°$ and $112\frac{1}{2}°$.

(iv) $\frac{2}{3}\pi$.   (v) $14°$ nearly.   (vi) $1\cdot19$, $2\cdot72$, $4\cdot92$.

(vii) $1\cdot16$, $3\cdot28$, $4\cdot95$.   (viii) $\pm\cdot82$.   (ix) $\cdot64$.

Examples XVII(a).   [Pages 221-224]

4. $\sqrt{2135}$ ft.

14. $17° 27' 30''$.

18. $18° 26' 5\cdot8''$ nearly.

Examples XVII(b).   [Pages 231-233]

1. $-\cos\left(a + \dfrac{\pi}{2n}\right)\Big/ \sin\dfrac{\pi}{2n}$.

2. 0.

3. $\dfrac{\sin\{a + \frac{1}{2}(n-1)(a+\pi)\} \sin \frac{1}{2}n(a+\pi)}{\sin\frac{1}{2}(a+\pi)}$.

4. $\dfrac{n}{2} + \dfrac{\sin n\theta}{2\sin\theta}\cos(n+1)\theta$.

5. $\dfrac{1}{4}\left(\dfrac{3\sin^2 na}{\sin a} - \dfrac{\sin^2 3na}{\sin 3a}\right)$.

6. $(-1)^{n-1}\dfrac{\sin n\theta \sin(n+1)\theta}{2\cos\theta}$.

7. $\dfrac{3}{8}n - \dfrac{1}{2}\dfrac{\sin n\pi}{\sin a}\cos(n+1)a + \dfrac{1}{8}\cdot\dfrac{\sin 2na}{\sin 2a}\cos 2(n+1)a$.

8. $\cos\{\theta + \frac{1}{2}(n-1)(\theta + \frac{1}{2}\pi)\}\dfrac{\sin\frac{1}{2}n(\theta + \frac{1}{2}\pi)}{\sin\frac{1}{2}(\theta + \frac{1}{2}\pi)}$.

9. $\dfrac{1}{4\sin a}\left\{(n+1)\sin 2a - \sin 2(n+1)a)\right\}$.

10. $\dfrac{n}{2}\cos 2a + \dfrac{\cos 2(n+1)a \sin 2na}{2\sin 2a}$.

11. $\frac{1}{2}$.

12. 0.

13. 0.

14. $\sin na$.

16. $\operatorname{cosec} a\{\tan(n+1)a - \tan a\}$.

17. $\operatorname{cosec}\theta\{\cot\theta - \cot(n+1)\theta\}$.

18. $\frac{1}{2}\operatorname{cosec} a\{\tan(n+1)a - \tan a\}$.

19. $\cot\theta\{\cot\theta - \cot(n+1)\theta\} - n$.

20. $\cot a - 2^n\cot 2^n a$.

21. $\dfrac{1}{2}\cdot\dfrac{\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\sin\frac{1}{2}(n+3)\theta - \dfrac{1}{4}\cdot\dfrac{\sin n\theta}{\sin\theta}\sin(n+3)\theta$.

22. $\frac{1}{2}(\tan 3^n x - \tan x)$.

23. $\tan^{-1}\dfrac{n}{2+n}$.

24. $\tan^{-1}\dfrac{n}{n+1}$.

25. $\tan^{-1}\dfrac{n}{n+1}$.

26. $\dfrac{1}{2^{n-1}}\cot\dfrac{x}{2^{n-1}} - 2\cot 2x$.

27. $\frac{1}{4}\left[\frac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x}\cos \frac{1}{2}(n+3)x(1+2\cos 2x)+\frac{\sin \frac{1}{2}nx}{\sin \frac{1}{2}x}\cos \frac{1}{2}(n+3)x\right].$

28. $\frac{(n+1)\cos n\theta-n\cos(n+1)\theta-1}{2(1-\cos \theta)}.$ **29.** (a) $\frac{1}{2}n(n+1).$ (b) $n^2.$

30. $\cot x \tan (n+1)x-(n+1)$; $\frac{1}{3}n(n+1)(n+2).$

Examples XVII(c). [Pages 236-237]

1. $(a^2-b^2)^2=ab.$ **2.** $a^2\{(x+b)^2+y^2\}=(x^2+y^2-b^2)^2.$

3. $(x+3y)^2=xy^2(x+2y).$ **4.** $a^2+b^2=1+b^{\frac{2}{3}}-b^{\frac{4}{3}}.$

5. $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2.$ **6.** $(x^2+y^2+2ax)^2=4a^2(x^2+y^2).$

7. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4^{\frac{2}{3}}.$ **8.** $x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1.$ **9.** $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$

10. $\frac{2x}{a}=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-3\right).$ **11.** $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$

12. $\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}=2.$ **13.** $x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1.$

14. $3a-2b=a^2.$ **15.** $\frac{x^2}{a}+\frac{y^2}{b}=a+b.$ **16.** $x^2+y^2-2\cos \alpha=2.$

17. $ab=(b-a)\tan \alpha.$ **18.** $a+b=2ab.$ **19.** $(ab-c)(a^2+b^2)=2ab.$

Miscellaneous Examples III [Pages 254-263]

1. $\frac{4}{15}\pi,\frac{1}{3}\pi,\frac{2}{5}\pi.$ **2.** $40°, 60°, 80°, \frac{2}{9}\pi, \frac{1}{3}\pi, \frac{4}{9}\pi.$ **3.** $90°, 22\frac{1}{2}°, 67\frac{1}{2}°.$

4. $1:36.$ **5.** $6\frac{4}{11}$ degrees.   **6.** $54°\ 46'\ 54·5''.$

26. $\frac{(t^2+1)(t^4+1)}{t^3}.$ **27.** $1.$ **28.** $\frac{7}{25}.$ **29.** $\frac{7}{13}.$

30. $x=\frac{1}{4}\pi,\ y=\frac{1}{6}\pi.$ **31.** (i) Possible.   (ii) Impossible.   (iii) Impossible.
      (iv) Possible.

32. (i) $a^2+b^2=2.$ (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$ (iii) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$

38. $341·5$ ft. approximately.   **39.** 125 yds.   **40.** $178·2$ ft.

41. $30°.$ **42.** $273·2$ ft.   **43.** 30 miles per hour.

44. height $=22·5$ ft., distance $\doteq 38·97$ ft.   **45.** $1, \frac{12}{13}.$

46. $\frac{27}{533}, \frac{14}{533}.$ **47.** $\frac{2mn}{m^2-n^2}.$ **48.** $-\frac{23}{63}.$ **49.** $\frac{24}{25}, -\frac{24}{7}, \frac{3}{5}\sqrt{5}.$

50. $\frac{7}{10}\sqrt{2}.$ **51.** $5+2\sqrt{6}.$ **52.** $-\frac{2}{3}\frac{6}{5}.$

84. $(2n+1)\frac{\pi}{2},$ or, $(2n+1)\pi\pm\frac{\pi}{8},$ or, $n\pi+(-1)^n\frac{\pi}{6}.$ **99.** $107·2$ ft.

HIGHER SECONDARY EXAMINATION PAPERS
(B. O. S. E., W. B.)

1. (a) If A, B and $A+B$ be all positive and acute, prove geometrically $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

(b) Show that $\cos 6° \cos 42° \cos 66° \cos 78° = \frac{1}{16}$.

2. (a) Solve : $\tan^2 x + \cot^2 x = 2$.

(b) Solve : $\sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}$.

3. (a) If $\alpha + \beta = \gamma$, show that

$$\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \gamma = \sin^2 \gamma.$$

(b) If $\dfrac{\tan (\alpha + \beta - \gamma)}{\tan (\alpha - \beta + \gamma)} = \dfrac{\tan \gamma}{\tan \beta}$, prove that either $\sin (\beta - \gamma) = 0$

or, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

4. (a) If in a triangle

$$(a^2 + b^2) \sin (A - B) = (a^2 - b^2) \sin (A + B),$$

prove that the triangle is either isosceles or right-angled.

(b) Find the greatest angle of a triangle whose sides are 5, 6, 7 having given $\log 6 = ·7781513$, $L \cos 39° \ 14' = 9·8890644$, diff. for $60'' = ·00011032$.

5. (a) Draw the graph of $y = \sin x + \cos x$ from $x = 0$ to $x = 2\pi$ havin given :

	10°	20°	30°	40°	50°	60°	70°	80°
sin x	·17	·34	·50	·64	·77	·87	·94	·98

(b) At each end of a horizontal line of length $2a$, the angular elevation of a certain peak is θ and that at the middle point is ϕ. Prove that the vertical height of the peak is

$$\frac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta) \sin (\phi - \theta)}}.$$

1. (a) If A, B, $A-B$ be positive acute angles, prove geometrically that

$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

(b) If $A+B+C = \dfrac{\pi}{2}$, prove that

$$\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C.$$

2. (a) Solve: $2\sin^2\theta + 3\cos\theta = 0$, $(0° < \theta < 360°)$.

(b) If $\sec\theta - \operatorname{cosec}\theta = \dfrac{4}{3}$, prove that

$$\theta = \frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right).$$

3. (a) In a triangle ABC, prove that

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

(b) Two sides of a triangle are 80 and 100 feet and the included angle is 60°; find the other angles having given

$$\log 3 = \text{·}47712, \ L\tan 10°\ 53'\ 36'' = 9\text{·}28432.$$

4. (a) Prove that

$$\cos^2 A + \cos^2\left(A+\frac{\pi}{3}\right) + \cos^2\left(A-\frac{\pi}{3}\right) = \frac{3}{2}.$$

(b) Prove that

$$2\cot^{-1}5 + \cot^{-1}7 + 2\cot^{-1}8 = \frac{\pi}{4}.$$

5. (a) Draw the graph of $\cos x - \sin x$ between $x = -\pi$ and $x = \pi$.

(b) A man standing on the bank of a river, observes that the angle subtended by a tower on the other bank just opposite to him is 60°. When he moves 60 ft. away from the bank in line with the tower he finds the angle subtended to be 30°; find the height of the tower and the breadth of the river.

1. (a) If C, D and $C+D$ be all positive acute angles, prove geometrically: $\sin(C+D) = \sin C \cos D + \cos C \sin D$.

(b) If $A + B + C = \pi$ and $\sin \left(A + \dfrac{C}{2} \right) = n \sin \dfrac{\cdot}{\cdot}$

prove that $\tan \dfrac{A}{2} \cdot \tan \dfrac{B}{2} = \dfrac{n-1}{n+1}$

2. (a) Solve :—

$\sin \theta + \sqrt{3} \cos \theta = \sqrt{2}$

(b) Show that,

$4(\cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5}) = \pi$

3. (a) If $\tan \dfrac{\theta}{2} = \sqrt{\dfrac{1-e}{1+e}} \tan \dfrac{\phi}{2}$,

show that $\cos \phi = \dfrac{\cos \theta - e}{1 - e \cos \theta}$

(b) Prove that, $\cos 36° = \frac{1}{4}(\sqrt{5} + 1)$

4. (a) Show that in any triangle ABC,

$a \cos A + b \cos B + c \cos C$

$= 4R \sin A \cdot \sin B \cdot \sin C$.

where R is the circumradius of the triangle,

(b) The sides of a triangle are 2, 3 and 4. Find the greatest angle of the triangle, having given :

$\log 2 = ·3010300$, $\log 3 = ·4771213$,

$L \tan 52°14' = 10·1108395$,

$L \tan 62°15' = 10·1111004$.

5. (a) Draw the graph of $y = 2 \sin x° + \cos x°$, for values of x between -30 to $+60$, having given :

x :	10	20	30	40	50	60	70	80
$\cos x°$:	·98	·94	·87	·77	·64	·50	·34	·17

6. (b) From the top A of a building, the angles of depression of the top B and bottom C of a lamp-post BC of height h and standing on the same horizontal plane are found to be a and β. Find the height of the building and show that $AB = \dfrac{h \cos \beta}{\sin (\beta - a)}$

GAUHATI UNIVERSITY INTER PAPERS

1. (a) Write down the expressions for $\sin (A + B)$ and $\cos (A + B)$ in terms of the trigonometrical ratios of A and B; hence deduce that for $\tan (A + B)$.

(b) Solve : $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$ (Give general values).

2. (a) If $A + B + C = \pi$, prove that
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

(b) Show that
$$\tan^{-1} \sqrt{\frac{a(a + b + c)}{bc}} + \tan^{-1} \sqrt{\frac{b(a + b + c)}{ca}}$$
$$+ \tan^{-1} \sqrt{\frac{c(a + b + c)}{ab}}.$$

3. (a) In any triangle prove that
$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

(b) The angles of a triangle, are in A. P., and the sides containing the mean angle are $\sqrt{3} + 1$ and $\sqrt{3} - 1$; find the other angles and the third side.

4. (a) Draw a neat graph of $\cos x$ in $-\pi < x < \pi$.

(b) Show that

(i) $\tan A = \dfrac{\sin 2A}{1 + \cos 2A}$;

(ii) $\tan A + \cot A = 2 \operatorname{cosec} 2A$.

1. (a) (i) Calculate $\sin 22\frac{1}{2}°$.

(ii) Express $2 \sin 11\theta \sin \theta$ as the difference of two terms.

(iii) If $\tan A = \frac{4}{5}$, $\tan B = \frac{1}{8}$, find the smallest positive value of $(A + B)$.

(b) If $A + B + C = \pi$, prove that
$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

2. (a) In a triangle ABC, prove that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

If $a = 3cm.$, $b = 4cm.$, $c = 5cm.$, find $\tan \frac{C}{2}$.

(b) Solve : $\sin \theta + \sin 7\theta = \sin 4\theta$ (give general values).

3. (a) Express $A \cos n\theta + B \sin n\theta$ in the form $R \cos(n\theta - x)$ and give the values of R and x. State the amplitude and period of

$$3 \cos \theta + 4 \sin \theta.$$

(b) Show that

$$\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2} + \tan^{-1} 1 = \frac{\pi}{2}.$$

4. *Either*, (a) Tabulate the values of $\sin x + \cos x$ as x varies from 0 to 2π at intervals of $\frac{\pi}{4}$.

Or, Find the area of $\triangle ABC$, given that $a = 13$, $b = 14$, $c = 15$.

(b) Prove that

(i) $\cos A = \dfrac{1 - \tan^2 \dfrac{A}{2}}{1 + \tan^2 \dfrac{A}{2}}$

(ii) $\sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A.$

PATNA UNIVERSITY QUESTIONS

1. (a) Show that $\cos^2 A \cos 3A + \sin^2 A \sin 3A = \cos^3 2A$.

   (b) If $x \sin^2\theta + y \cos^2\theta = \sin \theta \cos \theta$, and $x \sin \theta - y \cos \theta = 0$, show that $x^2 + y^2 = 1$.

2. (a) Establish the formula

$$\cos B - \cos A = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

   (b) Prove that $\cos^2 A + \cos^2 B + \cos^2 C - 2 \cos A \cos B \cos C = 1$, if $A + B = C$.

3. (a) Prove that in a triangle, $\cos \frac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$.

   (b) $\dfrac{a^2 \sin (B-C)}{\sin B + \sin C} + \dfrac{b^2 \sin (C-A)}{\sin C + \sin A} + \dfrac{c^2 \sin (A-B)}{\sin A + \sin B} = 0.$

4. (a) Draw the graph of $y = \sin x + \cos x$ as x ranges from 0 to π.

   (b) Prove that $\cot A + \cot B + \cot C = \cot A \cot B \cot C$, if $A + B + C = \frac{1}{2}\pi$.

5. (a) Prove that $\log_b n = \log_a n \times \log_b a$.

   (b) To determine the breadth AB of a canal an observer places himself at C in the straight line AB produced through C, and then walks 100 yards at right angles to the line. He then finds that AB and BC subtend angles 15° and 25° at his eyes. Find the breadth of the canal, given $L \cos 25° = 9{\cdot}9572757$; $L \cos 40° = 9{\cdot}8842540$; $L \cos 75° = 9{\cdot}4129962$; $\log 37279 = 4{\cdot}5714643$; $\log 3728 = 3{\cdot}5714759$.

1. (a) Evaluate $\sin 18°$.

   (b) If $\sec (\phi + a) + \sec (\phi - a) = 2 \sec \phi$, prove that

$$\cos \phi = \sqrt{2} \cos \frac{a}{2}.$$

2. (a) If $A + B + C = \pi$, prove that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi-A}{4} \cos \frac{\pi-B}{4} \cos \frac{\pi-C}{4}.$$

   (b) Draw the graph of $y = \tan x$ from $x = 0$ to $x = 2\pi$.

3. In a triangle ABC, prove that

(i) $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$.

(ii) $\cot A$, $\cot B$, $\cot C$ are in A.P., if a^2, b^2, c^2 are in A.P.

4. Two sides of a triangle are in the ratio of 9 to 7, and the included angle is $64° 12'$; find the other angles, having given $\log 2 = \cdot 3010300$, $L \tan 57° 54' = 10\cdot2025255$, $L \tan 11° 16' = 9\cdot2993216$, $L \tan 11° 17' = 9\cdot2999804$.

5. A flagstaff PN stands vertically on level ground. A base XY is measured at right angles to XN. the points X, Y, N being in the same horizontal plane, and the angle PXN and PYN are found to be α and β respectively. Prove that the height of the flagstaff is

$$\sqrt{\dfrac{\sin \alpha \sin \beta}{\sin (\alpha - \beta) \sin (\alpha + \beta)}} \, XY.$$

1. (a) If $\dfrac{x}{\tan (\theta + \alpha)} = \dfrac{y}{\tan (\theta + \beta)} = \dfrac{z}{\tan (\theta + \gamma)}$, prove that

$$\dfrac{x+y}{x-y} \sin^2 (\alpha - \beta) + \dfrac{y+z}{y-z} \sin^2 (\beta - \gamma) + \dfrac{z+x}{z-x} \sin^2 (\gamma - \alpha) = 0.$$

(b) Prove that $\cot A + \cot (60° + A) + \cot (120° + A) = 3 \cot 3A$.

2. (a) If $A + B + C = 180°$ and $\sin \left(A + \dfrac{C}{2} \right) = n \sin \dfrac{C}{2}$, show that

$$\tan \dfrac{A}{2} \tan \dfrac{B}{2} = \dfrac{n-1}{n+1}.$$

(b) If $A + B + C = 180°$, prove that

$$\sin^2 A + \sin^2 B + \sin^2 C - 2 \cos A \cos B \cos C = 2.$$

3. In a triangle, Prove that

(i) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$.

(ii) $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$.

4. (a) Draw the graph of $y = \sin x$ from $x = 0$ to $x = \pi$, and from the graph find the angle whose sine is $\cdot 7$.

(b) If $a = 70$, $b = 35$, $C = 36° 52' 12''$, find the other angles, having given $\log 3 = \cdot 4771213$, $L \cot 18° 26' 6'' = 10\cdot4771213$.

5. A flagstaff is on the top of a tower which stands on a level plane. At a certain point in the plane the tower subtends an angle α, and the flagstaff an angle β. At another points 'a' ft. nearer the base of the tower, the flagstaff again subtends the angle β. Show that the height of the tower is $\dfrac{a \tan \alpha}{1 - \tan \alpha \tan (\alpha + \beta)}$.

ALLAHABAD UNIVERSITY QUESTIONS

1. (a) Prove that $\cot \dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan \dfrac{A+B}{2} \cot \dfrac{A-B}{2}$.

(b) Prove that $\cot \left(\dfrac{\pi}{4} + \theta \right) \times \cot \left(\dfrac{\pi}{4} - \theta \right) = 1$.

2. (a) Prove that $\dfrac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta$.

(b) Solve the equation $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$.

3. (a) Prove that $\tan \left(45° + \dfrac{A}{2} \right) = \sqrt{\dfrac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$.

(b) If $A + B + C = 180°$, prove that

$$\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$$

4. (a) Prove the formula $r = \dfrac{S}{s}$, where the letters have their usual meanings.

(b) Prove that in any triangle, $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

5. The sides of a triangle are 32, 40, and 66 feet. Find the angle opposite the greatest side, having given

$$\log 3 = ·47712, \ \log 69 = 1·83885, \ \log 37 = 1·56820,$$

$$\log 29 = 1·46240, \ L \cot 66° \ 10' = 9·64517, \ L \cot 66° \ 20' = 9·64175.$$

6. (a) If $A + B + C = 2R$, prove that

$$\sin (S - A) + \sin (S - B) + \sin (S - C) - \sin S$$

$$= 4 \sin \dfrac{A}{2} \sin \dfrac{B}{2} \sin \dfrac{C}{2}.$$

(b) In any triangle ABC, prove that

$$(b + c - a) \left(\cot \dfrac{B}{2} + \cot \dfrac{C}{2} \right) = 2a \cot \dfrac{A}{2}.$$

7. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ, and that at the middle point it is ϕ. Prove that the vertical height of the peak is

$$\dfrac{a \sin \theta \sin \phi}{\sqrt{\sin (\phi + \theta) \sin (\phi - \theta)}}.$$

1. (a) Prove that

$$\frac{\cos A + \cos B}{\cos B - \cos A} = \cot \frac{A+B}{2} \cot \frac{A-B}{2}.$$

(b) Solve the equation $4 \cos^2\theta + \sqrt{3} = 2(\sqrt{3}+1) \cos \theta$.

2. (a) If $\tan a = \frac{4}{9}$, and $\tan \beta = \frac{1}{11}$, prove that

$$a + \beta = \frac{\pi}{4}.$$

(b) Show that

$$2 \cos \frac{\pi}{13} + \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0.$$

3. (a) Find the value of $\cos 36°$.

(b) Prove that

$$\cot A = \frac{1}{2}\left(\cot \frac{A}{2} - \tan \frac{A}{2}\right).$$

4. (a) Prove the formula $R = \frac{abc}{4S}$, where the letters have their usual meanings.

(b) Prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}.$$

5. In the triangle ABC, $b=16$, $c=25$ and the angle $B=33° 15'$. Find the remaining angles if

$$\log 2 = \cdot 30103,$$
$$L \sin 33° 15' = 9\cdot7390129,$$
$$L \sin 58° 56' = 9\cdot9327616, \quad \text{and}$$
$$L \sin 58° 57' = 9\cdot9328876.$$

6. (a) If $A+B+C = 2S$, prove that

$$\sin (S-A) \sin (S-B) + \sin S \sin (S-C) = \sin A \sin B.$$

(b) If the sides of a triangle be in arithmetical progression, prove that so also are the cotangents of half the angles.

7. At a distance a from the foot A of a tower AB, of known height b, a flagstaff BC on the top of the tower and the tower both subtend equal angles. Find the height of the flagstaff.

1. (a) Prove that
$$\frac{\sin 7A - \sin 5A}{\cos 5A + \cos 7A} = \tan A.$$

   (b) Prove that
$$\sin A = 2 \tan \frac{A}{2} \bigg/ \left(1 + \tan^2 \frac{A}{2}\right).$$

2. (a) Prove that $\tan 2A = (\sec 2A + 1) \sqrt{\sec^2 A - 1}$.

   (b) Solve the equation $\sin \theta + \sin 3\theta + \sin 5\theta = 0$.

3. (a) Find the value of $\sin 22\frac{1}{2}°$.

   (b) Prove that
$$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}} \sin A.$$

4. (a) Prove the formula
$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$
where the letters have their usual meanings.

   (b) Prove that
$$r_1 r_2 r_3 = r^3 \cot^2 \frac{A}{2} \cot^2 \frac{B}{2} \cot^2 \frac{C}{2}.$$

5. If the lengths of the greatest and least sides of a triangle be 24 and 16 feet repectively and the angle between them be 60°, find the length of the third side and the remaining angles, given $\log 2 = ·30103$, $\log 3 = ·4771213$, and $L \tan 19° 6' = 9·5394217$, diff. for $1' = 4084$.

6. (a) If $A + B + C = 2S$, prove that
$$\cos^2 S + \cos^2 (S - A) + \cos^2 (S - B) + \cos^2 (S - C)$$
$$= 2 + 2 \cos A \cos B \cos C.$$

   (b) In any triangle ABC, prove that
$$a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$$

7. A tower, more than 100 feet high, consists of two parts, the lower being one-third of the whole. At a point in a horizontal plane through the foot of the tower and 40 feet from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the tower.

BENARES HINDU UNIVERSITY QUESTIONS

1. Expand the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{vmatrix}$$

Hence or otherwise prove that for any $\triangle ABC$,

$$a \sin (B-C) + b \sin (C-A) + c \sin (A-B) = 0.$$

2. (a) Prove that

$$2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}.$$

(b) Solve the equation $\tan \theta = \cot \theta$.

3. Prove the following :

(i) $\dfrac{\sec 8A - 1}{\sec 4A - 1} = \tan 8A \cot 2A.$

(ii) $\tan 6° \tan 42° \tan 66° \tan 78° = 1.$

(iii) $\sin (A+B) \sin (A-B) = \sin^2 A - \sin^2 B.$

4. (a) Obtain the radius of the in-circle of a triangle in terms of the lengths of its sides.

(b) If r and R denote respectively the radii of the inscribed and circumscribed circles of a triangle ABC, prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2rR}.$$

5. (a) In any triangle ABC, prove that

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

(b) In a triangle ABC, $a = 540$ yards, $b = 420$ yards, $\angle C = 50° 6'$. Find the unknown angles, having given—

$$\log 2 = \cdot30103$$
$$L \tan 26° \quad 3' = 9\cdot6891430$$
$$L \tan 14° \quad 20' = 9\cdot4074189$$
$$L \tan 14° \quad 21' = 9\cdot4079453$$

6. The angle of elevation of a cloud from a point h feet above a lake is a and the angle of depression of its reflection in the lake is β. Prove that the height of the cloud above the lake is

$$h \, \frac{\sin (\beta + a)}{\sin (\beta - a)}.$$

7. (a) Prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$, where $A + B + C = 180°$.

(b) Solve the equation
$$\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}.$$

1. (a) If an angle subtended by an arc of length l at the centre of a circle of radius r be taken as a unit, and three angles $A°$, B^g and C radians expressed in that unit be x, y, z respectively, show that

$$x : y : z = \frac{A\pi}{18} : \frac{B\pi}{20} : 10C.$$

(b) Prove that
$$\cos (A - B) = \cos A \cos B + \sin A \sin B.$$

2. Solve :

(i) $\tan (\pi \cot \theta) = \cot (\pi \tan \theta)$.

(ii) $\sin 2\theta = \cot 3\theta$.

(iii) $\sqrt{3} \sin \theta - \cos \theta = \sqrt{2}$.

3. Prove that

(i) $\tan (\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$
$$= \cot (\cot^{-1} x + \cot^{-1} y + \cot^{-1} z).$$

(ii) $\left(\dfrac{1}{r} - \dfrac{1}{r_1} \right)\left(\dfrac{1}{r} - \dfrac{1}{r_2} \right)\left(\dfrac{1}{r} - \dfrac{1}{r_3} \right) = \dfrac{16R}{r^2(a+b+c)^2}$,

where the symbols r, r_1, r_2, r_3 and R have their usual meanings.

4. (a) Prove that

$$16 \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = 1.$$

(b) In any triangle ABC, prove that
$$a^2 (\cos^2 B - \cos^2 C) + b^2 (\cos^2 C - \cos^2 A)$$
$$+ c^2 (\cos^2 A - \cos^2 B) = 0.$$

5. (*a*) Prove that
$$\sin 4A = 4 \sin A \cos^3 A - 4 \cos A \sin^3 A.$$

(*b*) If the angles of a triangle are in A.P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the length of the third side and the angles, given—

$$\log 2 = \cdot 3010300,$$
$$\log 3 = \cdot 4771213$$
$$L \tan 19° \ 6' = 9\cdot 5394287,$$
$$\text{diff. for } 1' = 4084.$$

1.) Prove the following :

(i) $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A.$

(ii) $\sin 20° \sin 40° \sin 60° \sin 80° = \tfrac{3}{16}.$

(iii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C,$ where $A + B + C = 180°.$

2. (*a*) Prove that $\sin^{-1} \tfrac{3}{5} + \sin^{-1} \tfrac{8}{17} = \sin^{-1} \tfrac{77}{85}.$

(*b*) Solve $\tan \theta + \tan 2\theta + \sqrt{3} \tan \theta \tan 2\theta = \sqrt{3}.$

3. (*a*) Discuss the 'ambiguous case' in the solution of triangles.

(*b*) In the ambiguous case, given a, b and A, prove that the difference between the two values of c is $2\sqrt{a^2 - b^2 \sin^2 A}.$

4. (*a*) In any triangle ABC, prove that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}.$

(*b*) Prove that $r_1 + r_2 + r_3 - r = 4R.$

5. An object is observed at three points A, B, C lying in a horizontal straight line which passes directly underneath the object. The angular elevation at B is twice that at A, and at C it is three times that at A. If $AB = a$, $BC = b$, show that the height of the object is $\dfrac{a}{2b} \sqrt{(a+b)(3b-a)}.$

TABLES OF LOGARITHMS, NATURAL SINES, NATURAL TANGENTS, LOGARITHMIC SINES, LOGARITHMIC TANGENTS ETC.

TABLE I

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9		1	2	3	4	5	6	7	8	9
											Mean Differences									
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743		42	83	125	166	208	248	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555		38	76	114	152	190	227	265	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059		35	70	105	140	175	209	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301		33	65	97	129	163	198	225	258	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319		30	60	90	120	150	180	210	240	270
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140		28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789		26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285		25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646		23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885		22	45	67	89	111	134	156	178	201
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015		21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044		20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984		19	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840		19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620		18	36	53	71	89	107	124	142	160
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330		17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975		16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560		16	32	47	63	79	95	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090		15	30	46	61	76	91	106	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567		15	29	44	59	74	88	103	118	132

TABLE I] LOGARITHMS OF NUMBERS III

Proportional parts (differences):

9	8	7	6	5	4	3	2	1
129	114	100	86	72	57	43	29	14
125	110	97	83	69	55	42	28	14
121	107	94	80	67	54	40	27	13
117	104	91	78	65	52	39	26	13
113	101	88	76	63	50	38	25	13
110	98	86	73	61	49	37	24	12
107	95	83	71	60	48	36	24	12
104	93	81	70	58	46	35	23	12
102	90	79	68	57	45	34	23	11
99	88	77	66	55	44	33	22	11
97	86	75	64	54	43	32	21	11
94	84	73	63	52	42	31	21	10
92	82	71	61	51	41	31	20	10
90	80	70	60	50	40	30	20	10
88	78	68	59	49	39	29	20	10
86	76	67	57	48	38	29	19	10
84	75	65	56	47	37	28	19	9
82	73	64	55	46	37	27	18	9
80	71	62	53	45	36	27	18	9
79	70	61	53	44	35	26	18	9
77	69	60	52	43	34	26	17	9
76	67	59	51	42	34	25	17	8
75	66	58	50	42	33	25	17	8
73	66	57	49	41	32	24	16	8
72	64	56	48	40	32	24	16	8

Logarithms:

	0	1	2	3	4	5	6	7	8	9
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
45	65321	65418	65514	65610	65706	65801	65896	65992	66097	66181
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	63	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38	45	53	60	68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	45	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	57	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35	42	49	56	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34	41	48	55	62
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	39	45	52	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31	37	44	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	18	25	31	37	43	49	55
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30	36	42	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	18	23	29	35	41	47	52

TABLE I]   LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9									
											9	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	52	46	40	35	29	23	17	12	6
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	51	46	40	34	29	23	17	11	6
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	50	45	39	34	28	23	17	11	6
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	50	44	39	33	28	22	17	11	5
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	49	44	38	33	27	22	16	11	5
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	49	43	38	32	27	21	16	11	5
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	48	43	37	32	26	21	16	11	5
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	47	42	37	31	26	21	16	11	5
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	47	42	36	31	26	21	16	10	5
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	46	41	36	30	25	20	15	10	5
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	46	41	36	30	25	20	15	10	5
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	45	40	35	30	25	20	15	10	5
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	45	40	35	30	25	20	15	10	5
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	44	39	35	29	24	20	15	10	5
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	44	39	34	29	24	19	14	10	5
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	43	38	34	29	24	19	14	10	5
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	43	38	33	29	24	19	14	9	5
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	43	38	33	28	24	19	14	9	5
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	42	38	33	28	23	19	14	9	5
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	42	37	33	28	23	18	14	9	5
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	41	36	32	27	23	18	14	9	5
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	41	36	32	27	23	18	13	9	5
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	40	36	31	27	22	18	13	9	4
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	40	35	31	26	22	18	13	9	4
99	99564	99607	99651	99695	99739	99783	99826	99870	99913	99957	39	35	30	26	22	17	13	9	4

TABLE —
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
												Mean Differences					
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01454	0·01745	89°	29	58	87	116	145	175	204	233	262
1°	·01745	·02036	·02327	·02618	·02908	·03199	·03490	88°	29	58	87	116	145	175	204	233	262
2°	·03490	·03781	·04071	·04362	·04653	·04943	·05234	87°	29	58	87	116	145	175	204	233	262
3°	·05234	·05524	·05814	·06105	·06395	·06685	·06976	86°	29	58	87	116	145	174	203	232	261
4°	·06976	·07266	·07556	·07846	·08136	·08426	·08716	85°	29	58	87	116	145	174	203	232	261
5°	0·08716	·09005	·09295	·09585	·09874	·10164	0·10453	84°	29	58	87	116	145	174	203	232	261
6°	·10453	·10743	·11031	·11320	·11609	·11898	·12187	83°	29	58	87	116	145	174	203	232	261
7°	·12187	·12476	·12764	·13053	·13341	·13629	·13917	82°	29	58	87	116	145	173	202	231	260
8°	·13917	·14205	·14493	·14781	·15069	·15356	·15643	81°	29	58	86	115	144	173	202	230	259
9°	·15643	·15931	·16218	·16505	·16792	·17078	·17365	80°	29	57	86	115	144	172	201	230	258
10°	0·17365	·17651	·17937	·18224	·18509	·18795	0·19081	79°	29	57	86	115	144	172	201	229	258
11°	·19081	·19366	·19652	·19937	·20222	·20507	·20791	78°	29	57	86	114	143	171	200	228	257
12°	·20791	·21076	·21360	·21644	·21928	·22212	·22495	77°	28	57	85	114	142	170	199	227	256
13°	·22495	·22778	·23062	·23345	·23627	·23910	·24192	76°	28	57	85	113	141	170	198	226	255
14°	·24192	·24474	·24756	·25038	·25320	·25601	·25882	75°	28	56	85	113	141	169	197	226	254
15°	0·25882	·26163	·26443	·26724	·27004	·27284	0·27564	74°	28	56	84	112	140	168	196	224	252
16°	·27564	·27843	·28123	·28402	·28680	·28959	·29237	73°	28	56	84	112	140	167	195	223	251
17°	·29237	·29515	·29793	·30071	·30348	·30625	·30902	72°	28	56	83	111	139	166	194	222	250
18°	·30902	·31178	·31454	·31730	·32006	·32282	·32557	71°	28	55	83	110	138	166	193	221	248
19°	·32557	·32832	·33106	·33381	·33655	·33929	·34202	70°	27	55	82	110	137	164	192	219	247

°	0'	10'	20'	30'	40'	50'	60'	°	1'	2'	3'	4'	5'	6'	7'	8'	9'
20°	0·34202	·34475	·34748	·35021	·35293	·35565	0·35837	69°	27	55	82	109	137	164	191	218	246
21°	·35837	·36108	·36379	·36650	·36921	·37191	·37461	68°	27	54	81	108	136	163	190	217	244
22°	·37461	·37730	·37999	·38268	·38537	·38805	·39073	67°	27	54	81	108	135	161	188	215	242
23°	·39073	·39341	·39608	·39875	·40142	·40408	·40674	66°	27	53	80	107	134	160	187	214	240
24°	·40674	·40939	·41204	·41469	·41734	·41998	·42262	65°	27	53	80	106	133	159	186	212	238
25°	0·42262	·42525	·42788	·43051	·43313	·43575	0·43837	64°	26	52	79	105	131	157	184	210	236
26°	·43837	·44098	·44359	·44620	·44880	·45140	·45399	63°	26	52	78	104	130	156	182	208	234
27°	·45399	·45658	·45917	·46175	·46433	·46690	·46947	62°	26	52	77	103	129	155	181	206	232
28°	·46947	·47204	·47460	·47716	·47971	·48226	·48481	61°	26	51	77	102	128	154	179	204	230
29°	·48481	·48735	·48989	·49242	·49495	·49748	·50000	60°	25	51	76	101	127	152	177	202	228
30°	0·50000	·50252	·50503	·50754	·51004	·51254	0·51504	59°	25	50	75	100	125	150	175	200	225
31°	·51504	·51753	·52002	·52250	·52498	·52745	·52992	58°	25	50	74	99	124	149	174	198	223
32°	·52992	·53238	·53484	·53730	·53975	·54220	·54464	57°	25	49	74	99	123	147	172	196	221
33°	·54464	·54708	·54951	·55194	·55436	·55678	·55919	56°	24	49	73	97	122	146	170	194	219
34°	·55919	·56160	·56401	·56641	·56880	·57119	·57358	55°	24	48	72	96	120	144	168	192	216
35°	0·57358	·57596	·57833	·58070	·58307	·58543	0·58779	54°	24	47	71	95	119	142	166	190	213
36°	·58779	·59014	·59248	·59482	·59716	·59949	·60182	53°	23	47	70	94	117	140	164	187	211
37°	·60182	·60414	·60645	·60876	·61107	·61337	·61566	52°	23	46	70	92	116	139	162	185	208
38°	·61566	·61795	·62024	·62251	·62479	·62706	·62932	51°	23	46	68	91	114	137	159	182	205
39°	·62932	·63158	·63383	·63608	·63832	·64056	·64279	50°	22	45	67	90	112	135	157	179	202
40°	0·64279	·64501	·64723	·64945	·65166	·65386	0·65606	49°	23	44	66	88	111	133	155	177	199
41°	·65606	·65825	·66044	·66262	·66480	·66697	·66913	48°	22	44	65	87	109	131	153	174	196
42°	·66913	·67129	·67344	·67559	·67773	·67987	·68200	47°	21	43	64	86	107	129	150	172	193
43°	·68200	·68412	·68624	·68835	·69046	·69256	·69466	46°	21	42	63	84	106	127	147	169	190
44°	·69466	·69675	·69883	·70091	·70298	·70505	·70711	45°	21	42	62	83	104	124	145	166	187
	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'

NATURAL COSINES

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	0·70711	0·70916	0·71121	0·71325	0·71529	0·71732	0·71934	44°	20	41	61	82	102	122	143	163	184
46°	·71934	·72136	·72337	·72537	·72737	·72937	·73135	43°	20	40	60	80	100	120	140	160	18C
47°	·73135	·73333	·73531	·73728	·73924	·74120	·74314	42°	20	39	59	78	98	118	138	157	177
48°	·74314	·74509	·74703	·74896	·75088	·75280	·75471	41°	19	39	58	77	96	116	135	154	173
49°	·75471	·75661	·75851	·76041	·76229	·76417	·76604	40°	19	38	57	76	95	113	132	151	170
50°	0·76604	0·76791	0·76977	0·77152	0·77347	0·77531	0·77715	39°	19	37	56	74	93	111	130	148	167
51°	·77715	·77897	·78079	·78251	·78442	·78622	·78801	38°	18	36	54	72	91	109	127	145	163
52°	·78801	·78980	·79158	·79335	·79512	·79688	·79864	37°	18	35	53	71	89	106	·124	142	159
53°	·79864	·80038	·80212	·80386	·80558	·80730	·80902	36°	17	35	52	69	87	104	121	138	156
54°	·80902	·81072	·81242	·81412	·81580	·81748	·81915	35°	17	34	51	68	85	101	118	135	152
55°	0·81915	0·82082	0·82248	0·82413	0·82577	0·82741	0·82904	34°	16	33	49	66	82	99	115	132	148
56°	·82904	·83066	·83228	·83389	·83549	·83708	·83867	33°	16	32	48	64	80	96	112	128	144
57°	·83867	·84025	·84182	·84339	·84495	·84650	·84805	32°	16	31	47	63	78	94	110	125	141
58°	·84805	·84959	·85112	·85264	·85416	·85567	·85717	31°	15	30	46	61	76	91	106	122	137
59°	·85717	·85866	·86015	·86163	·86310	·86457	·86603	30°	15	30	44	59	74	89	103	118	133
60°	0·86603	0·86748	0·86892	0·87036	0·87178	0·87321	0·87462	29°	14	29	43	57	72	86	100	114	129
61°	·87462	·87603	·87743	·87882	·88020	·88158	·88295	28°	14	28	42	55	69	83	97	111	125
62°	·88295	·88431	·88566	·88701	·88835	·88968	·89101	27°	13	27	40	54	67	81	94	108	121
63°	·89101	·89232	·89363	·89493	·89623	·89752	·89879	26°	13	26	39	52	65	78	91	104	117
64°	·89879	·90007	·90133	·90259	·90383	·90507	·90631	25°	13	25	38	50	63	75	88	100	113

Mean Differences

TABLE II]   NATURAL SINES AND COSINES   IX

	60'	50'	40'	30'	20'	10'	0'		1'	2'	3'	4'	5'	6'	7'	8'	9'
65°	0·90631	·90753	·90875	·90996	·91116	·91236	·91355	24°	12	24	36	48	60	72	84	96	108
66°	0·91355	·91472	·91590	·91706	·91822	·91936	·92050	23°	12	23	35	46	58	70	81	93	104
67°	0·92050	·92164	·92276	·92388	·92499	·92609	·92718	22°	11	22	33	45	56	67	78	89	100
68°	0·92718	·92827	·92935	·93042	·93148	·93253	·93358	21°	11	21	32	43	53	64	75	85	96
69°	0·93358	·93462	·93565	·93667	·93769	·93869	·93969	20°	10	20	31	41	51	61	71	81	92
70°	0·93969	·94068	·94167	·94264	·94361	·94457	·94552	19°	10	19	29	39	49	58	68	78	87
71°	0·94552	·94646	·94740	·94832	·94924	·95015	·95106	18°	9	18	28	37	46	55	64	74	83
72°	0·95106	·95195	·95284	·95372	·95459	·95545	·95630	17°	9	18	26	35	44	52	61	70	79
73°	0·95630	·95715	·95799	·95882	·95964	·96046	·96126	16°	8	17	25	33	41	50	58	66	74
74°	0·96126	·96206	·96285	·96363	·96440	·96517	·96593	15°	8	16	23	31	39	47	54	62	70
75°	0·96593	·96667	·96742	·96815	·96887	·96959	·97030	14°	7	15	22	29	36	44	51	58	65
76°	0·97030	·97100	·97169	·97237	·97304	·97371	·97437	13°	7	14	20	27	34	41	47	54	61
77°	0·97437	·97502	·97566	·97630	·97692	·97754	·97815	12°	6	13	19	25	32	38	44	50	57
78°	0·97815	·97875	·97934	·97992	·98050	·98107	·98163	11°	6	12	17	23	29	35	41	46	52
79°	0·98163	·98218	·98272	·98325	·98378	·98430	·98481	10°	5	11	16	21	27	32	37	42	48
80°	0·98481	·98531	·98580	·98629	·98676	·98723	·98769	9°	5	10	14	19	24	29	34	38	43
81°	0·98769	·98814	·98858	·98902	·98944	·98986	·99027	8°	4	9	13	17	22	26	30	34	39
82°	0·99027	·99067	·99106	·99144	·99182	·99219	·99255	7°	4	8	11	15	19	23	27	30	34
83°	0·99255	·99290	·99324	·99357	·99390	·99421	·99452	6°	3	7	10	13	17	20	23	26	30
84°	0·99452	·99482	·99511	·99540	·99567	·99594	·99619	5°	3	6	8	11	14	17	20	22	25
85°	0·99619	·99644	·99668	·99692	·99714	·99736	·99756	4°	2	5	7	9	12	14	16	18	21
86°	0·99756	·99776	·99795	·99813	·99831	·99847	·99863	3°	2	4	5	7	9	11	13	14	16
87°	0·99863	·99878	·99892	·99905	·99917	·99929	·99939	2°	1	3	4	5	7	8	9	10	12
88°	0·99939	·99949	·99958	·99966	·99973	·99979	·99985	1°									
89°	0·99985	·99989	·99993	·99996	·99998	·99999	1·00000	0°									
90°	1·00000																

NATURAL COSINES

TABLE III
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	Mean Differences 4′	5′	6′	7′	8′	9′
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01455	0·01746	89°	29	58	87	116	146	175	204	233	262
1°	·01746	·02037	·02328	·02619	·02910	·03201	·03492	88°	29	58	87	116	146	175	204	233	262
2°	·03492	·03783	·04075	·04366	·04658	·04949	·05241	87°	29	58	87	116	146	1·5	204	233	262
3°	·05241	·05533	·05824	·06116	·06408	·06700	·06993	86°	29	58	88	117	146	175	204	234	263
4°	·06993	·07285	·07578	·07870	·08163	·08456	·08749	85°	29	59	88	117	146	175	204	234	263
5°	0·08749	0·09042	0·09335	0·09629	0·09923	0·10216	0·10510	84°	29	59	88	118	147	176	206	235	265
6°	·10510	·10805	·11099	·11394	·11688	·11983	·12278	83°	29	59	88	118	147	176	206	235	265
7°	·12278	·12574	·12869	·13165	·13461	·13758	·14054	82°	30	59	89	118	148	178	207	237	266
8°	·14054	·14351	·14648	·14945	·15243	·15540	·15838	81°	30	59	89	119	149	178	208	238	267
9°	·15838	·16137	·16435	·16734	·17033	·17333	·17633	80°	30	60	90	120	150	179	209	239	269
10°	0·17633	0·17933	0·18233	0·18534	0·18835	0·19136	0·19438	79°	30	60	90	120	151	181	211	241	271
11°	·19438	·19740	·20042	·20345	·20648	·20952	·21256	78°	30	61	91	121	152	182	212	242	273
12°	·21256	·21560	·21864	·22169	·22475	·22781	·23087	77°	31	61	92	122	153	183	214	244	275
13°	·23087	·23393	·23700	·24008	·24316	·24624	·24933	76°	31	62	92	123	154	185	216	246	277
14°	·24933	·25242	·25552	·25862	·26172	·26483	·26795	75°	31	62	93	124	155	186	217	248	279
15°	0·26795	0·27107	0·27419	0·27732	0·28046	0·28360	0·28675	74°	31	63	94	125	157	188	219	250	282
16°	·28675	·28990	·29305	·29621	·29938	·30255	·30573	73°	32	63	95	126	159	190	221	253	285
17°	·30573	·30891	·31210	·31530	·31850	·32171	·32492	72°	32	64	96	128	160	192	224	256	288
18°	·32492	·32814	·33136	·33460	·33783	·34108	·34433	71°	32	65	97	129	162	194	226	259	291
19°	·34433	·34758	·35085	·35412	·35740	·36068	·36397	70°	33	65	98	131	164	196	229	262	294

TABLE III] NATURAL TANGENTS XI

NATURAL TANGENTS

(For tangents read the degrees in the left‑hand column and the minutes 0′–60′ across the top; for cotangents read the degrees in the right‑hand column and the minutes 60′–0′.)

°	0′	10′	20′	30′	40′	50′	60′	°
20°	0·36397	0·36727	0·37057	0·37388	0·37720	0·38053	0·38386	69°
21°	0·38386	0·38721	0·39055	0·39391	0·39727	0·40065	0·40403	68°
22°	0·40403	0·40741	0·41081	0·41421	0·41763	0·42105	0·42447	67°
23°	0·42447	0·42791	0·43136	0·43481	0·43828	0·44175	0·44523	66°
24°	0·44523	0·44872	0·45222	0·45573	0·45924	0·46277	0·46631	65°
25°	0·46631	0·46985	0·47341	0·47698	0·48055	0·48414	0·48773	64°
26°	0·48773	0·49134	0·49495	0·49858	0·50222	0·50587	0·50953	63°
27°	0·50953	0·51320	0·51688	0·52057	0·52427	0·52798	0·53171	62°
28°	0·53171	0·53545	0·53920	0·54296	0·54673	0·55051	0·55431	61°
29°	0·55431	0·55812	0·56194	0·56577	0·56962	0·57348	0·57735	60°
30°	0·57735	0·58124	0·58513	0·58905	0·59297	0·59691	0·60086	59°
31°	0·60086	0·60483	0·60881	0·61280	0·61681	0·62083	0·62487	58°
32°	0·62487	0·62892	0·63299	0·63707	0·64117	0·64528	0·64941	57°
33°	0·64941	0·65355	0·65771	0·66189	0·66608	0·67028	0·67451	56°
34°	0·67451	0·67875	0·68301	0·68728	0·69157	0·69588	0·70021	55°
35°	0·70021	0·70455	0·70891	0·71329	0·71769	0·72211	0·72654	54°
36°	0·72654	0·73100	0·73547	0·73996	0·74447	0·74900	0·75355	53°
37°	0·75355	0·75812	0·76272	0·76733	0·77196	0·77661	0·78129	52°
38°	0·78129	0·78598	0·79070	0·79544	0·80020	0·80498	0·80978	51°
39°	0·80978	0·81461	0·81946	0·82434	0·82923	0·83415	0·83910	50°
40°	0·83910	0·84407	0·84906	0·85408	0·85912	0·86419	0·86929	49°
41°	0·86929	0·87441	0·87955	0·88473	0·88992	0·89515	0·90040	48°
42°	0·90040	0·90569	0·91099	0·91633	0·92170	0·92709	0·93252	47°
43°	0·93252	0·93797	0·94345	0·94896	0·95451	0·96008	0·96569	46°
44°	0·96569	0·97133	0·97700	0·98270	0·98843	0·99420	1·00000	45°
	60′	50′	40′	30′	20′	10′	0′	

Mean Differences

°	1′	2′	3′	4′	5′	6′	7′	8′	9′
20°	33	66	100	133	166	199	232	265	298
21°	34	67	101	134	168	202	236	269	302
22°	34	68	102	136	170	205	239	273	306
23°	35	69	104	138	173	208	242	277	311
24°	35	70	105	140	176	211	246	281	316
25°	36	71	107	143	179	214	250	286	321
26°	36	73	109	145	182	218	254	291	327
27°	37	74	111	148	185	222	259	296	333
28°	38	75	113	151	189	226	264	302	339
29°	38	77	115	154	192	230	269	307	346
30°	39	78	118	157	196	235	274	313	353
31°	40	80	120	160	200	240	280	320	360
32°	41	82	123	164	205	245	286	327	368
33°	42	84	126	167	209	251	293	334	376
34°	43	86	128	171	214	257	300	342	385
35°	44	88	132	176	220	263	307	351	395
36°	45	90	135	180	225	270	315	360	405
37°	46	92	139	185	231	277	324	370	416
38°	48	95	143	190	238	285	333	380	428
39°	49	98	147	196	245	293	342	391	440
40°	50	101	151	201	252	302	352	402	453
41°	52	104	156	208	260	311	363	415	467
42°	54	107	161	214	268	321	375	429	482
43°	55	111	166	221	277	332	387	442	498
44°	57	114	172	229	286	343	400	457	515

NATURAL COTANGENTS

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
												Mean Differences					
45°	1·00000	1·00583	1·01170	1·01761	1·02355	1·02952	1·03553	44°	59	118	178	237	296	355	414	474	533
46°	·03553	·04158	·04766	·05378	·05994	·06613	·07237	43°	61	123	184	246	307	368	430	491	553
47°	·07237	·07864	·08496	·09131	·09770	·10414	·11061	42°	64	127	191	255	319	382	446	510	573
48°	·11061	·11713	·12369	·13029	·13694	·14363	·15037	41°	66	132	199	265	332	397	463	530	596
49°	·15037	·15715	·16398	·17085	·17777	·18474	·19175	40°	69	138	207	276	345	413	482	552	620
50°	1·19175	·19882	·20593	·21310	·22031	·22758	1·23490	39°	72	144	216	288	360	431	503	575	647
51°	·23490	·24227	·24969	·25717	·26471	·27230	·27994	38°	75	150	225	300	376	451	526	601	676
52°	·27994	·28764	·29541	·30323	·31110	·31904	·32704	37°	78	157	235	314	392	471	549	628	707
53°	·32704	·33511	·34323	·35142	·35968	·36800	·37638	36°	82	164	247	329	411	493	576	658	740
54°	·37638	·38484	·39336	·40195	·41061	·41934	·42815	35°	86	172	259	345	431	517	603	690	776
55°	1·42815	1·43703	1·44598	1·45501	1·46411	1·47380	1·48256	34°	91	181	272	363	453	544	634	725	816
56°	·48256	·49190	·50133	·51084	·52043	·53010	·53987	33°	96	191	287	382	478	573	669	764	860
57°	·53987	·54972	·55966	·56969	·57981	·59002	·60033	32°	101	201	302	403	504	604	705	806	907
58°	·60033	·61074	·62125	·63185	·64256	·65337	·66428	31°	107	213	320	426	533	639	746	852	959
59°	·66428	·67530	·68643	·69766	·70901	·72047	·73205	30°	118	226	339	451	565	677	790	903	1016
60°	1·7321	1·7487	1·7556	1·7675	1·7796	1·7917	1·8040	29°	13	24	36	48	60	72	84	96	108
61°	1·8040	1·8165	1·8291	1·8418	1·8546	1·8676	1·8807	28°	13	25	38	51	64	77	89	102	115
62°	1·8807	1·8940	1·9074	1·9210	1·9347	1·9486	1·9626	27°	14	27	41	54	68	82	95	109	122
63°	1·9626	1·9768	1·9912	2·0057	2·0204	2·0353	2·0503	26°	15	29	44	58	73	88	102	117	131
64°	2·0503	2·0655	2·0809	2·0965	2·1123	2·1283	2·1445	25°	16	31	47	63	79	94	110	126	141

TABLE III]   NATURAL TANGENTS   XIII

NATURAL TANGENTS

tan °	0′	10′	20′	30′	40′	50′	60′	cot °
65°	2·1445	2·1609	2·1775	2·1943	2·2113	2·2286	2·2460	24°
66°	2·2460	2·2637	2·2817	2·2998	2·3183	2·3369	2·3559	23°
67°	2·3559	2·3750	2·3945	2·4142	2·4342	2·4545	2·4751	22°
68°	2·4751	2·4960	2·5172	2·5386	2·5605	2·5826	2·6051	21°
69°	2·6051	2·6279	2·6511	2·6746	2·6985	2·7228	2·7475	20°
70°	2·7475	2·7725	2·7980	2·8239	2·8502	2·8770	2·9042	19°
71°	2·9042	2·9319	2·9600	2·9887	3·0178	3·0475	3·0777	18°
72°	3·0777	3·1084	3·1397	3·1716	3·2041	3·2371	3·2709	17°
73°	3·2709	3·3052	3·3402	3·3759	3·4124	3·4495	3·4874	16°
74°	3·4874	3·5261	3·5656	3·6059	3·6470	3·6891	3·7321	15°
75°	3·7321	3·7760	3·8208	3·8667	3·9136	3·9617	4·0108	14°
76°	4·0108	4·0611	4·1126	4·1653	4·2193	4·2747	4·3315	13°
77°	4·3315	4·3897	4·4414	4·5107	4·5786	4·6383	4·7046	12°
78°	4·7046	4·7729	4·8430	4·9152	4·9894	5·0658	5·1446	11°
79°	5·1446	5·2257	5·3093	5·3955	5·4845	5·5764	5·6713	10°
80°	5·6713	5·7694	5·8708	5·9758	6·0844	6·1970	6·3138	9°
81°	6·3138	6·4348	6·5606	6·6912	6·8269	6·9682	7·1154	8°
82°	7·1154	7·2687	7·4287	7·5958	7·7704	7·9530	8·1443	7°
83°	8·1443	8·3450	8·5555	8·7769	9·0098	9·2553	9·6144	6°
84°	9·6144	9·7882	10·0780	10·3854	10·7119	11·0594	11·4301	5°
85°	11·4301	11·8262	12·2505	12·7062	13·1969	13·7267	14·3007	4°
86°	14·3007	14·9244	15·6048	16·3499	17·1693	18·0750	19·0811	3°
87°	19·0811	20·2056	21·4704	22·9038	24·5419	26·4316	28·6363	2°
88°	28·6363	31·2416	34·3678	38·1885	42·9641	49·1039	57·2900	1°
89°	57·2900	68·7501	85·9398	114·589	171·885	343·774	+∞	0°
90°	+∞							

Proportional parts (differences)

°	1′	2′	3′	4′	5′	6′	7′	8′	9′
65°	17	34	51	68	85	101	118	135	152
66°	18	37	55	73	92	110	128	146	165
67°	20	40	60	80	100	119	139	159	179
68°	22	43	65	87	109	130	152	174	195
69°	24	47	71	95	119	142	166	190	213
70°	26	52	78	104	131	157	183	209	235
71°	29	58	87	116	145	174	202	231	260
72°	32	64	97	129	161	193	225	258	290
73°	36	72	108	144	181	216	253	289	325
74°	41	81	122	163	204	244	285	326	366
75°	46	93	139	185	232	278	325	371	418
76°	53	107	160	214	267	320	374	427	481
77°	62	124	186	248	311	373	435	497	559
78°	73	146	220	293	366	439	512	586	659
79°	88	175	263	350	438	526	613	701	788

The differences change very rapidly here so that they cannot be tabulated.

The cotangent of a small angle of x′ or the tangent of 90° − x′ is very nearly equal to 3437·7 divided by x.

NATURAL COTANGENTS

Cotangents are read from the same grid using the degree labels 24° … 0° and the minute headings reversed (60′ … 0′).

Table IV
LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	−∞	7·46373	7·76475	7·94084	8·06578	8·16268	8·24186	89°									
1°	8·24186	8·30879	8·36678	8·41792	8·46366	8·50504	8·54282	88°									
2°	8·54282	8·57757	8·60973	8·63968	8·66769	8·69400	8·71880	87°									
3°	8·71880	8·74226	8·76451	8·78568	8·80585	8·82513	8·84358	86°									
4°	8·84358	8·86128	8·87829	8·89464	8·91040	8·92561	8·94030	85°									
5°	8·94030	8·95450	8·96825	8·98157	8·99450	9·00704	9·01923	84°									
6°	9·01923	9·03109	9·04262	9·05386	9·06481	9·07548	9·08589	83°									
7°	9·08589	9·09606	9·10599	9·11570	9·12519	9·13447	9·14356	82°	96	192	288	384	480	576	672	768	864
8°	9·14356	9·15245	9·16116	9·16970	9·17807	9·18628	9·19433	81°	85	169	254	338	423	507	592	676	761
9°	9·19433	9·20233	9·20999	9·21761	9·22509	9·23244	9·23967	80°	76	151	227	302	378	453	529	604	680
10°	9·23967	9·24677	9·25376	9·26063	9·26739	9·27405	9·28060	79°	68	136	204	272	341	409	477	545	613
11°	9·28060	9·28705	9·29340	9·29966	9·30582	9·31189	9·31788	78°	62	124	186	248	310	373	435	497	559
12°	9·31788	9·32378	9·32960	9·33534	9·34100	9·34658	9·35209	77°	57	114	171	228	285	342	399	456	513
13°	9·35209	9·35752	9·36289	9·36819	9·37341	9·37858	9·38368	76°	53	105	158	210	263	316	368	421	473
14°	9·38368	9·38871	9·39369	9·39860	9·40346	9·40825	9·41300	75°	49	98	147	195	244	293	342	391	440
15°	9·41300	9·41768	9·42232	9·42690	9·43143	9·43591	9·44034	74°	46	91	137	182	228	273	319	364	410
16°	9·44034	9·44472	9·44905	9·45334	9·45758	9·46178	9·46594	73°	43	85	128	171	213	256	299	341	384
17°	9·46594	9·47005	9·47411	9·47814	9·48213	9·48607	9·48998	72°	40	80	120	160	201	241	281	321	361
18°	9·48998	9·49385	9·49768	9·50148	9·50523	9·50896	9·51264	71°	38	76	113	151	189	227	264	302	340
19°	9·51264	9·51629	9·51991	9·52350	9·52705	9·53056	9·53405	70°	36	71	107	149	179	214	250	285	321

Differences vary so rapidly here that tabulation is impossible. For small angles of x minutes $\log \sin x'$ or $\log \cos (90° - x') = \log x + 4\cdot46373$.

TABLE IV]   LOGARITHMIC SINES   XV

Sin°	9′	8′	7′	6′	5′	4′	3′	2′	1′	Cos°	0′	10′	20′	30′	40′	50′	60′
20°	304	270	237	203	169	135	101	68	34	69°	9·55433	·55102	·54769	·54493	·54093	·53751	·53405
21°	289	257	225	193	161	128	96	64	32	68°	·57358	·57044	·56727	·56408	·56085	·55761	·55433
22°	275	244	214	183	153	122	92	61	31	67°	·59188	·58889	·58588	·58284	·57978	·57669	·57358
23°	262	233	204	174	146	116	87	58	29	66°	·60931	·60646	·60359	·60070	·59778	·59484	·59188
24°	250	222	195	166	139	111	83	56	28	65°	·62595	·62323	·62049	·61773	·61494	·61214	·60931
25°	239	212	186	159	133	106	80	53	27	64°	9·64184	·63924	·63662	·63398	·63133	·62865	·62595
26°	229	203	178	152	127	102	76	51	25	63°	·65705	·65456	·65205	·64953	·64698	·64442	·64184
27°	219	194	170	146	122	97	73	49	24	62°	·67161	·66942	·66682	·66441	·66197	·65952	·65705
28°	210	186	163	140	117	93	70	47	23	61°	·68557	·68329	·68098	·67866	·67633	·67398	·67161
29°	201	179	156	134	112	89	67	45	22	60°	·69897	·69677	·69456	·69234	·69010	·68784	·68557
30°	193	172	150	129	107	86	65	43	22	59°	9·71184	·70973	·70761	·70547	·70332	·70115	·69897
31°	185	165	144	124	103	82	62	41	21	58°	·72421	·72218	·72014	·71809	·71602	·71393	·71184
32°	178	159	139	119	99	79	59	40	20	57°	·73611	·73416	·73219	·73022	·72823	·72622	·72421
33°	172	153	134	115	96	76	57	38	19	56°	·74756	·74568	·74379	·74189	·73997	·73805	·73611
34°	165	147	129	110	92	74	55	37	18	55°	·75859	·75678	·75496	·75313	·75128	·74943	·74756
35°	159	142	124	106	89	71	53	35	18	54°	9·76922	·76747	·76572	·76305	·76218	·76030	·75859
36°	154	137	120	103	86	69	51	34	17	53°	·77946	·77773	·77609	·77430	·77263	·77085	·76922
37°	149	132	116	98	83	66	50	33	17	52°	·78934	·78772	·78609	·78445	·78280	·78113	·77946
38°	143	127	110	95	80	64	48	32	16	51°	·79887	·79731	·79573	·79415	·79256	·79095	·78934
39°	138	123	105	92	77	62	46	31	15	50°	·80807	·80656	·80504	·80351	·80197	·80043	·79887
40°	133	118	104	89	73	59	44	30	15	49°	9·81694	·81519	·81402	·81254	·81106	·80957	·80807
41°	129	114	100	86	72	57	43	29	14	48°	·82551	·82410	·82269	·82126	·81953	·81839	·81694
42°	124	110	97	83	69	55	41	28	14	47°	·83378	·83240	·83106	·82968	·82826	·82691	·82551
43°	120	106	93	80	67	53	40	27	13	46°	·84177	·84046	·83914	·83781	·83648	·83513	·83378
44°	115	102	90	77	64	51	38	26	13	45°	·84949	·84822	·84694	·84566	·84437	·84309	·84177
	9′	8′	7′	6′	5′	4′	3′	2′	1′		0′	10′	20′	30′	40′	50′	60′

LOGARITHMIC COSINES

LOGARITHMIC SINES

	0′	10′	20′	30′	40′	50′	60′		Mean Differences								
									1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	9·84949	9·85074	·85200	·85324	·85448	·85571	9·85693	44°	12	25	37	50	62	74	87	99	112
46°	·85693	·85815	·85936	·86056	·86176	·86294	·86413	43°	12	24	36	48	60	72	84	96	108
47°	·86413	·86530	·86647	·86763	·86879	·86998	·87107	42°	12	23	35	46	58	70	81	93	104
48°	·87107	·87221	·87334	·87446	·87557	·87663	·87778	41°	11	22	34	45	56	67	78	89	100
49°	·87778	·87887	·87996	·88105	·88212	·88319	·88425	40°	11	22	32	43	54	65	76	86	97
50°	9·88425	·88531	·88636	·88741	·88844	·88948	9·89050	39°	10	21	31	42	52	63	73	83	94
51°	·89050	·89152	·89254	·89354	·89455	·89554	·89653	38°	10	20	30	40	50	60	70	80	90
52°	·89653	·89752	·89849	·89947	·90043	·90139	·90235	37°	10	19	29	39	49	58	68	78	87
53°	·90235	·90330	·90424	·90518	·90611	·90704	·90796	36°	9	19	28	37	47	55	65	74	84
54°	·90796	·90837	·90978	·91069	·91158	·91241	·91336	35°	9	18	27	36	45	54	63	72	81
55°	9·91336	9·91425	·91512	·91599	·91686	·91772	9·91857	34°	9	17	26	35	44	52	61	70	78
56°	·91857	·91942	·92027	·92111	·92194	·92277	·92359	33°	8	17	25	34	42	50	59	67	76
57°	·92359	·92441	·92522	·92603	·92683	·92763	·92842	32°	8	16	24	32	41	49	57	65	73
58°	·92842	·92921	·92999	·93077	·93154	·93230	·93307	31°	8	16	23	31	39	47	55	62	70
59°	·93307	·93383	·93457	·93532	·93606	·93680	·93753	30°	8	15	23	30	37	45	52	60	67
60°	9·93753	9·93826	·93898	·93970	·94041	·94112	9·94182	29°	7	14	22	29	36	43	50	57	64
61°	·94182	·94252	·94321	·94390	·94458	·94526	·94593	28°	7	14	21	27	34	41	48	55	62
62°	·94593	·94660	·94727	·94793	·94858	·94923	·94988	27°	7	13	20	26	33	40	46	53	59
63°	·94988	·95052	·95116	·95179	·95242	·95304	·95366	26°	6	13	19	25	32	38	44	50	57
64°	·95366	·95427	·95488	·95549	·95609	·95668	·95728	25°	6	12	18	24	30	36	42	48	54

TABLE IV] LOGARITHMIC SINES XVII

LOGARITHMIC SINES (mantissas given with the characteristic; read minutes 60′ → 0′ left to right; the 0′ column is the whole‑degree value)

° (sin)	° (cos)	60′	50′	40′	30′	20′	10′	0′
65°	24°	9·96073	9·96017	9·95960	9·95902	9·95844	9·95786	9·95728
66°	23°	9·96403	9·96349	9·96294	9·96240	9·96185	9·96129	9·96073
67°	22°	9·96717	9·96665	9·96614	9·96562	9·96509	9·96456	9·96403
68°	21°	9·97015	9·96966	9·96917	9·96868	9·96818	9·96767	9·96717
69°	20°	9·97299	9·97252	9·97206	9·97159	9·97111	9·97063	9·97015
70°	19°	9·97567	9·97523	9·97479	9·97435	9·97390	9·97344	9·97299
71°	18°	9·97821	9·97779	9·97738	9·97696	9·97653	9·97610	9·97567
72°	17°	9·98060	9·98021	9·97982	9·97942	9·97902	9·97861	9·97821
73°	16°	9·98284	9·98248	9·98211	9·98174	9·98136	9·98098	9·98060
74°	15°	9·98494	9·98460	9·98426	9·98391	9·98356	9·98320	9·98284
75°	14°	9·98690	9·98659	9·98627	9·98594	9·98561	9·98528	9·98494
76°	13°	9·98872	9·98843	9·98813	9·98783	9·98753	9·98722	9·98690
77°	12°	9·99040	9·99013	9·98986	9·98958	9·98930	9·98901	9·98872
78°	11°	9·99195	9·99170	9·99145	9·99119	9·99093	9·99067	9·99040
79°	10°	9·99335	9·99313	9·99290	9·99267	9·99243	9·99219	9·99195
80°	9°	9·99462	9·99442	9·99421	9·99400	9·99379	9·99357	9·99335
81°	8°	9·99575	9·99557	9·99539	9·99520	9·99501	9·99482	9·99462
82°	7°	9·99675	9·99659	9·99643	9·99627	9·99610	9·99593	9·99575
83°	6°	9·99761	9·99748	9·99734	9·99720	9·99705	9·99690	9·99675
84°	5°	9·99834	9·99823	9·99812	9·99800	9·99787	9·99775	9·99761
85°	4°	9·99894	9·99885	9·99876	9·99866	9·99856	9·99845	9·99834
86°	3°	9·99940	9·99934	9·99926	9·99919	9·99911	9·99903	9·99894
87°	2°	9·99974	9·99969	9·99964	9·99959	9·99953	9·99947	9·99940
88°	1°	9·99993	9·99991	9·99988	9·99985	9·99982	9·99978	9·99974
89°	0°	10·00000	9·99999	9·99999	9·99998	9·99997	9·99995	9·99993
90°		10·00000	10·00000	10·00000	10·00000	10·00000	10·00000	10·00000

Proportional parts

°	9′	8′	7′	6′	5′	4′	3′	2′	1′
65°	52	46	40	35	29	23	17	12	6
66°	50	44	39	33	28	22	17	11	6
67°	47	42	36	31	26	21	16	10	5
68°	44	40	34	29	25	20	15	10	5
69°	42	38	33	28	24	19	14	10	5
70°	40	36	31	27	22	18	13	9	4
71°	38	34	30	26	21	17	13	9	4
72°	36	32	28	24	20	16	12	8	4
73°	34	30	26	22	19	15	11	8	4
74°	32	29	25	21	18	14	11	7	4
75°	30	26	23	20	17	13	10	7	3
76°	27	24	21	18	15	12	9	6	3
77°	25	22	20	17	14	11	8	6	3
78°	23	21	18	16	13	10	8	5	3
79°	21	19	16	14	12	9	7	5	2
80°	19	17	15	13	11	8	6	4	2
81°	17	15	13	11	10	8	6	4	2
82°	15	13	12	10	9	7	5	3	1
83°	13	12	10	9	7	6	4	3	1
84°	11	10	9	8	6	5	4	2	1
85°	9	8	7	6	5	4	3	2	1
86°	7	6	5	5	4	3	2	2	1
87°	5	4	4	3	3	2	2	1	1
88°	3	2	2	2	1	1	1	1	0

LOGARITHMIC COSINES

(bottom minute headers, left → right: 60′ 50′ 40′ 30′ 20′ 10′ 0′)

TABLE V
LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	−∞	7·46373	7·76476	7·94086	8·06581	8·16273	8·24192	**89°**	Differences vary so rapidly here that								
1°	8·24192	8·30888	8·36689	8·41807	8·46366	8·50527	8·54308	**88°**	tabulation is impossible.								
2°	8·54303	8·57788	8·61009	8·64009	8·66816	8·69453	8·71940	**87°**	For small angles of x minutes								
3°	8·71940	8·74292	8·76525	8·78649	8·80674	8·82610	8·84464	**86°**	log tan x or log cot $(90° - x')$								
4°	8·84464	8·86243	8·87953	8·89598	8·91185	8·92716	8·94195	**85°**	$= \log x + 7\cdot46373.$								
5°	8·94195	8·95627	8·97013	8·98358	8·99662	9·00930	9·02162	**84°**									
6°	9·02162	9·03361	9·04528	9·05666	9·06775	9·07858	9·08914	**83°**									
7°	9·08914	9·09917	9·10956	9·11943	9·12909	9·13854	9·14780	**82°**	98	195	293	391	488	586	684	782	879
8°	9·14780	9·15658	9·16577	9·17150	9·18306	9·19146	9·19971	**81°**	87	173	260	346	433	519	606	692	779
9°	9·19971	9·20782	9·21578	9·22361	9·23130	9·23887	9·24632	**80°**	78	155	233	310	388	466	543	621	698
10°	9·24632	9·25365	9·26086	9·26797	9·27496	9·28186	9·28865	**79°**	71	141	212	282	354	420	494	564	635
11°	·28865	·29535	·30195	·30843	·31489	·32122	·32747	**78°**	65	129	194	259	323	388	453	518	582
12°	·32747	·33365	·33974	·34576	·35170	·35757	·36336	**77°**	60	120	179	239	299	359	419	478	538
13°	·36336	·36909	·37476	·38085	·38589	·39136	·39677	**76°**	56	111	167	222	278	334	389	445	500
14°	·39677	·40312	·40743	·41266	·41784	·42297	·42805	**75°**	52	104	156	208	261	313	365	417	469
15°	9·42805	9·43308	9·43806	9·44299	9·44787	9·45271	9·45750	**74°**	49	98	147	196	245	294	343	392	442
16°	·45750	·46324	·46694	·47160	·47622	·48080	·48534	**73°**	46	93	139	186	232	278	325	371	418
17°	·48534	·48984	·49430	·49872	·50311	·50746	·51178	**72°**	44	88	132	176	220	264	308	352	396
18°	·51178	·51606	·52031	·52452	·52870	·53285	·53697	**71°**	42	84	126	168	210	252	294	336	378
19°	·53697	·54106	·54512	·54915	·55315	·55712	·56107	**70°**	40	80	121	160	201	241	281	321	362

(Mean Differences)

TABLE V] LOGARITHMIC TANGENTS XIX

Logarithmic Tangents

Left Deg	0′	10′	20′	30′	40′	50′	60′	Right Deg
20°	9·56107	·56498	·56887	·57274	·57658	·58039	·58418	69°
21°	9·58418	·58794	·59168	·59540	·59909	·60276	·60641	68°
22°	9·60641	·61004	·61364	·61722	·62079	·62433	·62785	67°
23°	9·62785	·63135	·63484	·63830	·64175	·64517	·64858	66°
24°	9·64858	·65197	·65535	·65870	·66204	·66537	·66867	65°
25°	9·66867	·67196	·67524	·67850	·68174	·68497	·68818	64°
26°	9·68818	·69138	·69457	·69774	·70089	·70404	·70717	63°
27°	9·70717	·71028	·71339	·71648	·71955	·72262	·72567	62°
28°	9·72567	·72872	·73175	·73476	·73777	·74077	·74375	61°
29°	9·74375	·74673	·74969	·75264	·75558	·75852	·76144	60°
30°	9·76144	·76435	·76725	·77015	·77303	·77591	·77877	59°
31°	9·77877	·78163	·78448	·78732	·79015	·79297	·79579	58°
32°	9·79579	·79860	·80140	·80419	·80697	·80975	·81252	57°
33°	9·81252	·81528	·81803	·82078	·82352	·82626	·82899	56°
34°	9·82899	·83171	·83442	·83713	·83984	·84254	·84523	55°
35°	9·84523	·84791	·85059	·85327	·85594	·85860	·86126	54°
36°	9·86126	·86392	·86656	·86921	·87185	·87449	·87711	53°
37°	9·87711	·87974	·88236	·88498	·88759	·89020	·89281	52°
38°	9·89281	·89541	·89801	·90061	·90320	·90578	·90837	51°
39°	9·90837	·91095	·91353	·91610	·91868	·92125	·92381	50°
40°	9·92381	·92638	·92894	·93150	·93406	·93661	·93916	49°
41°	9·93916	·94171	·94426	·94681	·94935	·95190	·95444	48°
42°	9·95444	·95698	·95952	·96205	·96459	·96712	·96966	47°
43°	9·96966	·97219	·97472	·97725	·97978	·98231	·98484	46°
44°	9·98484	·98737	·98989	·99242	·99495	·99747	10·00000	45°
	60′	50′	40′	30′	20′	10′		

Proportional Parts

Deg	9′	8′	7′	6′	5′	4′	3′	2′	1′
20°	347	308	270	231	193	154	116	77	39
21°	333	296	259	222	185	148	111	74	37
22°	322	286	250	214	179	143	107	72	36
23°	311	277	242	208	173	138	104	69	35
24°	302	268	235	201	168	134	101	67	34
25°	293	260	228	195	163	130	98	65	33
26°	284	253	221	190	158	126	95	63	32
27°	277	246	216	185	154	123	92	62	31
28°	271	241	211	181	151	120	90	60	30
29°	265	236	206	177	147	118	88	59	29
30°	260	231	202	173	144	116	87	58	29
31°	255	227	199	170	142	113	85	57	28
32°	251	223	195	167	140	112	84	56	28
33°	247	220	192	165	137	110	82	55	27
34°	244	217	190	162	135	108	81	54	27
35°	240	214	187	160	134	107	80	53	27
36°	238	212	185	158	132	106	79	53	26
37°	236	209	183	157	131	105	79	52	26
38°	234	208	182	156	130	104	78	52	26
39°	232	206	180	154	129	103	77	51	26
40°	230	205	179	154	128	102	77	51	26
41°	229	204	178	153	127	102	76	51	25
42°	228	203	177	152	127	101	76	51	25
43°	228	202	177	152	127	101	76	51	25
44°	226	202	177	152	127	101	76	51	25

LOGARITHMIC COTANGENTS

LOGARITHMIC TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
											Mean	Differences					
45°	10·00000	10·00253	10·00505	10·00758	10·01011	10·01263	10·01516	44°	25	51	76	101	127	152	177	202	228
46°	·01516	·01769	·02022	·02275	·02528	·02781	·03034	43°	25	51	76	101	127	152	177	202	228
47°	·03034	·03288	·03541	·03795	·04048	·04302	·04556	42°	25	51	76	101	127	152	177	203	228
48°	·04556	·04810	·05065	·05319	·05574	·05829	·06084	41°	25	51	76	102	127	153	178	204	229
49°	·06084	·06339	·06594	·06850	·07106	·07362	·07619	40°	26	51	77	102	128	154	179	205	230
50°	10·07619	10·07876	10·08132	10·08390	10·08647	10·08905	10·09163	39°	26	52	77	103	129	155	180	206	232
51°	·09163	·09422	·09680	·09959	·10199	·10459	·10719	38°	26	52	78	104	130	156	182	208	234
52°	·10719	·10980	·11241	·11502	·11764	·12026	·12289	37°	26	52	78	105	131	157	183	209	236
53°	·12289	·12552	·12815	·13079	·13344	·13608	·13874	36°	26	53	79	106	132	158	185	212	238
54°	·13874	·14140	·14406	·14673	·14941	·15209	·15477	35°	27	54	80	107	134	160	188	214	241
55°	10·15477	10·15746	10·16016	10·16287	10·16558	10·16829	10·17101	34°	27	54	81	108	136	162	190	217	244
56°	·17101	·17374	·17648	·17922	·18197	·18472	·18748	33°	28	55	83	110	137	165	192	220	247
57°	·18748	·19025	·19303	·19581	·19860	·20140	·20421	32°	28	56	84	112	139	167	195	223	251
58°	·20421	·20703	·20985	·21268	·21552	·21837	·22123	31°	28	57	85	113	142	170	198	227	255
59°	·22123	·22409	·22697	·22985	·23275	·23565	·23856	30°	29	58	87	116	144	173	202	231	260
60°	10·23856	10·24148	10·24442	10·24736	10·25031	10·25327	10·25625	29°	29	59	88	118	147	177	206	236	265
61°	·25625	·25923	·26223	·26524	·26825	·27128	·27433	28°	30	60	90	120	151	181	211	241	271
62°	·27433	·27738	·28045	·28352	·28661	·28972	·29283	27°	31	62	92	123	154	185	216	246	277
63°	·29283	·29596	·29911	·30226	·30543	·30862	·31182	26°	32	63	95	126	158	190	221	253	284
64°	·31182	·31503	·31826	·32150	·32476	·32804	·33133	25°	33	65	98	130	163	195	228	260	293

TABLE V]   LOGARITHMIC TANGENTS   XXI

Logarithmic Tangents

Deg	0′	10′	20′	30′	40′	50′	60′	Cot
65°	10·33133	·33463	·33796	·34180	·34465	·34808	·35142	24°
66°	·35142	·35488	·35825	·36170	·36516	·36865	·37215	23°
67°	·37215	·37567	·37921	·38278	·38636	·38996	·39359	22°
68°	·39359	·39724	·40091	·40460	·40832	·41206	·41582	21°
69°	·41582	·41961	·42342	·42726	·43113	·43502	·43898	20°
70°	10·43898	·44288	·44685	·45085	·45488	·45894	·46303	19°
71°	·46303	·46715	·47130	·47548	·47969	·48394	·48822	18°
72°	·48822	·49254	·49689	·50128	·50570	·51016	·51466	17°
73°	·51466	·51920	·52378	·52840	·53306	·53776	·54250	16°
74°	·54250	·54729	·55213	·55701	·56194	·56692	·57195	15°
75°	10·57195	·57703	·58216	·58734	·59258	·59788	·60323	14°
76°	·60323	·60864	·61411	·61965	·62524	·63091	·63664	13°
77°	·63664	·64243	·64830	·65424	·66026	·66635	·67253	12°
78°	·67253	·67878	·68511	·69154	·69805	·70465	·71135	11°
79°	·71135	·71814	·72504	·73203	·73914	·74635	·75363	10°
80°	10·75363	·76113	·76870	·77639	·78422	·79218	·80029	9°
81°	·80029	·80854	·81694	·82550	·83423	·84312	·85220	8°
82°	·85220	·86146	·87091	·88057	·89044	·90053	·91086	7°
83°	·91086	·92142	·93225	·94384	·95472	·96639	·97838	6°
84°	·97838	·99070	11·00338	11·01649	11·02987	11·04373	11·05805	5°
85°	11·05805	11·07284	11·08815	11·10403	11·12047	11·13757	11·15536	4°
86°	11·15536	11·17390	11·19926	11·21851	11·23475	11·25708	11·28060	3°
87°	11·28060	11·30347	11·33184	11·35991	11·38991	11·42212	11·45692	2°
88°	11·45692	11·49473	11·53615	11·58199	11·63311	11·69112	11·75808	1°
89°	11·75808	11·83727	11·93419	12·05914	12·23524	12·53627	+∞	0°
90°	+∞							

| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | |

LOGARITHMIC COTANGENTS

Proportional Parts

1′	2′	3′	4′	5′	6′	7′	8′	9′
34	67	101	134	168	201	235	268	302
35	69	104	138	173	208	242	277	311
36	72	107	143	179	214	251	286	322
37	74	111	148	185	222	259	296	333
39	77	116	154	193	231	270	308	347
40	80	121	160	201	241	281	321	362
42	84	126	168	210	252	294	336	378
44	88	132	176	220	264	308	352	396
46	93	139	186	232	278	325	371	418
49	98	147	196	245	294	343	392	442
52	104	156	208	261	313	365	417	469
56	111	167	222	278	334	389	445	500
60	120	179	239	299	359	419	478	538
65	129	194	259	323	388	453	518	582
71	141	212	282	354	420	494	564	635
78	155	233	310	388	466	543	621	698
87	173	260	346	433	519	606	692	779
98	195	293	391	488	586	684	782	879

Differences vary so rapidly here that tabulation is impossible.

SOME USEFUL CONSTANTS

One radian $= 57°\ 17'\ 45''$ nearly $= 206265''$;

$$\log 206265 = 5\text{·}3144255.$$

$$\pi = 3\text{·}14159265\ldots \qquad \frac{1}{\pi} = 0\text{·}31830989.$$

$$\sqrt{2} = 1\text{·}4142135\cdots \qquad \sqrt{3} = 1\text{·}7320508\ldots$$

$$\sqrt{5} = 2\text{·}2360679\ldots \qquad \sqrt{6} = 2\text{·}4494897\ldots$$

$$\sqrt{7} = 2\text{·}6457513\ldots \qquad \sqrt{8} = 2\text{·}8284271\ldots$$

$$\sqrt{10} = 3\text{·}1622776\ldots$$

SOME USEFUL LOGARITHMS

$$\log 2 = \text{·}30103 \qquad\qquad \log 3 = \text{·}47712$$

$$\log 5 = \text{·}69897 \qquad\qquad \log 7 = \text{·}84510$$

———